Line defects in the vanishing elastic constant limit of a three-dimensional Landau-de Gennes model

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Abstract

We consider the Landau-de Gennes variational model for nematic liquid crystals, in three-dimensional domains. We are interested in the asymptotic behaviour of minimizers as the elastic constant tends to zero. Assuming that the energy of minimizers is bounded by the logarithm of the elastic constant, there exists a relatively closed, 1-rectifiable set $\mathscr{S}_{\mathrm{line}}$ of finite length, such that minimizers converge to a locally harmonic map away from $\mathscr{S}_{\mathrm{line}}$. We provide sufficient conditions for the logarithmic energy bound to be satisfied. Finally, we show by an example that the limit map may have both point and line singularities.

Keywords. Landau-de Gennes model, Q-tensors, asymptotic analysis, topological singularities, line defects, rectifiable sets.
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1 Introduction

1.1 Variational theories for nematic liquid crystals

A nematic liquid crystal is matter in an intermediate state between liquid and crystalline solid. Molecules can flow as in a liquid, but they are oriented in an ordered way. As a result, the material is anisotropic with respect to optic and electromagnetic properties. Here, we restrict our attention to uniaxial nematics. These materials are composed by rod-shaped (sometimes, disk-shaped) molecules, with indistinguishable ends. The symmetry group of such a molecule is generated by rotations around the molecular axis, and the reflection symmetry which exchange the ends of the molecules. The word *nematic* was coined by Friedel, and originates from the line defects which are observed in these materials (see [29]):

> *I am going to use the term nematic (νήμα, thread) to describe the forms, bodies, phases, etc. of the second type... because of the linear discontinuities, which are twisted like threads, and which are one of their most prominent characteristics.*

In addition to line defects, also called *disclinations*, nematic media exhibit "hedgehog-like" point singularities. According to the topological theory of ordered media (see e.g. [48, 58, 60]), both kinds of defects are described by the homotopy groups of a certain manifold, which parametrizes the possible local configurations of the material.

Three main continuum theories for uniaxial nematic liquid crystals have drawn the attention of the mathematical community: the Oseen-Frank, the Ericksen and the Landau-de Gennes theories. In the

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Oseen-Frank theory [28], the material is modeled by a unit vector field $\mathbf{n} = \mathbf{n}(x) \in \mathbb{S}^2$, which represents the preferred direction of molecular alignment. The elastic energy, in the simplest setting, reduces to the Dirichlet functional

$$E(\mathbf{n}) := \frac{1}{2}\int_\Omega |\nabla \mathbf{n}|^2 \,, \tag{1}$$

where $\Omega \subseteq \mathbb{R}^3$ is the physical domain. In this case, least-energy configurations are but harmonic maps $\mathbf{n}\colon \Omega \to \mathbb{S}^2$. As such, minimizers have been widely studied in the literature (the reader is referred to e.g. [36] for a general review of this subject). Schoen and Uhlenbeck [55] proved that minimizers are smooth away from a discrete set of points singularities. Brezis, Coron and Lieb [17] investigated the precise shape of minimizers around a point defect x_0, and proved that

$$\mathbf{n}(x) \simeq \pm R\frac{x - x_0}{|x - x_0|} \qquad \text{for } |x - x_0| \ll 1, \tag{2}$$

where R is a rotation. These "hedgehog-like" point defects are associated with a non-trivial homotopy class of maps $\mathbf{n}\colon \partial B_r(x_0) \to \mathbb{S}^2$, i.e. a non-trivial element of $\pi_2(\mathbb{S}^2)$. Interesting results are also available for the full Oseen-Frank energy, which consists of various terms accounting for splay, twist and bend deformations. Hardt, Kinderlehrer and Lin [34] proved the existence of minimizers and partial regularity, i.e. regularity out of an exceptional set whose Hausdorff dimension is strictly less than 1. As for the local behaviour of minimizers around the defects, the picture is not as clear as for the Dirichlet energy (1), but at least the stability of "hedgehog-like" singularities such as (2) as been completely analyzed (see [41] and the references therein). However, the partial regularity result of [34] implies that the Oseen-Frank theory cannot account for line defects.

Ericksen theory is less restrictive, because it allows variable orientational order. Indeed, the configurations are described by a pair $(s, \mathbf{n}) \in \mathbb{R} \times \mathbb{S}^2$, where $\mathbf{n}$ is the preferred direction of molecular alignment and the scalar s measures the degree of ordering. In this theory, defects are identified by the condition $s = 0$, which correspond to complete disordered states. Under suitable assumptions, minimizers can exhibit line singularities and even planar discontinuities (see [43, Theorem 7.2]). Explicit examples were studied by Ambrosio and Virga [6] and Mizel, Roccato and Virga [49]. However, the Ericksen theory — as the Oseen-Franck theory — excludes configurations which might have physical reality. Ericksen himself was aware of this, since he presented his theory as a "kind of compromise" [27, p. 98] between physical intuition and mathematical simplicity. Indeed, both the Oseen-Franck and the Ericksen theory do not take into account the molecular symmetry, that is, the configurations represented by $\mathbf{n}$ and $-\mathbf{n}$ are physically indistinguishable. Moreover, these theories postulate that, at each point of the medium, there is at most one preferred direction of molecular orientation. Configurations for which such a preferred direction exists are called *uniaxial*, because they have one axis of rotational symmetry. If no preferred direction exists, the configuration is called *isotropic* (in the Ericksen theory, this corresponds to $s = 0$).

The Landau-de Gennes theory [25] allows for a rather complete description of the local behaviour of the medium, because it accounts for *biaxial*[1] configurations as well. A state is called biaxial when it has no axis of rotational symmetry, but three orthogonal axes of reflection symmetry instead. In a biaxial state, more preferred directions of molecular alignment coexist (see [51] for more details). What makes the Landau-de Gennes theory so rich is the choice of the order parameter space. Configurations are described by matrices (the so-called Q-tensors), which can be interpreted as renormalized second-order moments of a microscopic density, representing the distribution of molecules as a function of orientation.

In this paper, we aim at describing the generation of line defects for nematics in three-dimensional domains from a variational point of view, within the Landau-de Gennes theory. Two main simplifying assumptions are postulated here. First, we neglect the effect of external electromagnetic fields. Instead,

[1]Throughout this paper, "uniaxial" or "biaxial" refer to *arrangements* of molecules, not to the molecules themselves which are always assumed to be uniaxial.

to induce non-trivial behaviour in minimizers, we couple the problem with non-homogeneous Dirichlet boundary conditions (strong anchoring). Second, we adopt the one-constant approximation, that is we drop out several terms in the expression of the elastic energy, and we are left with the gradient-squared term only. These assumptions, which drastically reduce the technicality of the problem, are common in the mathematical literature on this subject (see e.g. [26, 30, 37, 38, 42, 47]). For the two-dimensional case, the analysis of the analogous problem is presented in [21, 32].

1.2 The Landau-de Gennes functional

As we mentioned before, the local configurations of the medium are described by Q-tensors, i.e. elements of

$$\mathbf{S}_0 := \left\{Q \in \mathrm{M}_3(\mathbb{R})\colon Q^\mathsf{T} = Q, \ \operatorname{tr} Q = 0\right\}.$$

This is a real linear space, of dimension five, which we endow with the scalar product $Q \cdot P := Q_{ij}P_{ij}$ (Einstein's convention is assumed). This choice of the configurations space can be justified as follows. At a microscopic scale, the distribution of molecules around a given point $x \in \Omega$, as a function of orientation, can be represented by a probability measure μ_x on the unit sphere $\mathbb{S}^2$. The measure μ_x satisfies to the condition $\mu_x(B) = \mu_x(-B)$ for all $B \in \mathscr{B}(\mathbb{S}^2)$, which accounts for the head-to-tail symmetry of the molecules. Then, the simplest meaningful way to condense the information conveyed by μ_x is to consider the second-order moment

$$Q = \int_{\mathbb{S}^2} \left(\mathbf{n}^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right) \mathrm{d}\mu_x(\mathbf{n}).$$

This quantity is renormalized, so that the isotropic state $\mu_x = \mathscr{H}^2 \llcorner \mathbb{S}^2$ corresponds to $Q = 0$. As a result, Q is a symmetric traceless matrix. (The interested reader is referred e.g. to [51] for further details).

The (simplified) Landau-de Gennes functional reads

$$E_\varepsilon(Q) := \int_\Omega \left\{\frac{1}{2}\left|\nabla Q\right|^2 + \frac{1}{\varepsilon^2} f(Q)\right\}, \tag{LG$_\varepsilon$}$$

where $Q\colon \Omega \to \mathbf{S}_0$ is the configuration of the medium, located in a bounded container $\Omega \subseteq \mathbb{R}^3$. The function f is the quartic Landau-de Gennes potential, defined by

$$f(Q) = k - \frac{a}{2}\operatorname{tr} Q^2 - \frac{b}{3}\operatorname{tr} Q^3 + \frac{c}{4}\left(\operatorname{tr} Q^2\right)^2 \qquad \text{for } Q \in \mathbf{S}_0. \tag{3}$$

This expression for f has been derived by a formal expansion in powers of Q. All the terms are invariant by rotations so that f is independent of the coordinate frame. This potential allows for multiple local minima, with a first-order isotropic-nematic phase transition (see [25, 59]). The positive parameters a, b and c depend on the material and the temperature (which is assumed to be uniform and constant), whereas k is just an additive constant, which plays no role in the minimization problem. The potential f is bounded from below, so we determine uniquely the value of k by requiring $\inf f = 0$. The parameter ε^2 is a material-dependent elastic constant, typically very small ($\varepsilon^2 \simeq 10^{-11}$ Jm^{-1}, as order of magnitude). For each $0 < \varepsilon < 1$, we assign a boundary datum $g_\varepsilon \in H^1(\partial\Omega, \mathbf{S}_0)$ and we restrict our attention to minimizers Q_ε of (LG$_\varepsilon$) in the class

$$H^1_{g_\varepsilon}(\Omega, \mathbf{S}_0) := \left\{Q \in H^1(\Omega, \mathbf{S}_0)\colon Q_{|\partial\Omega} = g_{\varepsilon|\partial\Omega} \text{ in the sense of traces}\right\}.$$

The set $\mathscr{N} := f^{-1}(0)$ is involved in the analysis of the problem. Indeed, when ε is very small the term $\varepsilon^{-2} f(Q)$ in (LG$_\varepsilon$) forces minimizers to take their values as close as possible to $\mathscr{N}$. The set $\mathscr{N}$ can be characterized as follows (see [47, Proposition 9]):

$$\mathscr{N} = \left\{s_* \left(\mathbf{n}^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right) : \mathbf{n} \in \mathbb{S}^2\right\}, \tag{4}$$

where the constant s_* is defined by

$$s_* = s_*(a,\, b,\, c) := \frac{1}{4c}\left(b + \sqrt{b^2 + 24ac}\right).$$

Thus, $\mathscr{N}$ is a smooth submanifold of $\mathbf{S}_0$, diffeomorphic to the real projective plane $\mathbb{RP}^2$, called *vacuum manifold*. The topology of $\mathscr{N}$ plays an important role, for a map $\Omega \to \mathscr{N}$ may encounter topological obstructions to regularity. Sources of obstruction are the homotopy groups $\pi_1(\mathscr{N}) \simeq \mathbb{Z}/2\mathbb{Z}$ and $\pi_2(\mathscr{N}) \simeq \mathbb{Z}$, which are associated with line and point singularities, respectively. There is a remarkable difference with the Oseen-Frank model at this level, for $\mathbb{S}^2$ is a simply connected manifold, so topological obstructions result from $\pi_2(\mathbb{S}^2)$ only. Despite this fact, a strong connection between the Oseen-Frank and Landau-de Gennes theories was established by Majumdar and Zarnescu. In their paper [47], they addressed the asymptotic analysis of minimizers of (LG$_\varepsilon$), in three-dimensional domains. Their results imply that, when Ω, $\partial\Omega$ are simply connected and $g_\varepsilon = g \in C^1(\partial\Omega,\, \mathscr{N})$, minimizers Q_ε of (LG$_\varepsilon$) converge in $H^1(\Omega,\, \mathbf{S}_0)$ to a map of the form

$$Q_0(x) = s_*\left(\mathbf{n}_0^{\otimes 2}(x) - \frac{1}{3}\,\mathrm{Id}\right)$$

where $\mathbf{n}_0 \in H^1(\Omega,\, \mathbb{S}^2)$ is a minimizer of (1). The convergence is locally uniform, away from singularities of Q_0. Also in this case, line defects do not appear in the limiting map, although point defects analogous to (2) might occur. Indeed, their assumptions on the domain and boundary datum are strong enough to guarantee the uniform energy bound

$$E_\varepsilon(Q_\varepsilon) \leq C \tag{5}$$

for an ε-independent constant C, and obtain H^1-compactness. In this paper, we work in the logarithmic energy regime

$$E_\varepsilon(Q_\varepsilon) \leq C\left(|\log\varepsilon| + 1\right), \tag{6}$$

which is compatible with singularities of codimension two, in the ε-vanishing limit.

There are analogies between the functional (LG$_\varepsilon$) and the Ginzburg-Landau energy for superconductors, which reduces to

$$E_\varepsilon^{\mathrm{GL}}(u) := \int_\Omega \left\{\frac{1}{2}\left|\nabla u\right|^2 + \frac{1}{4\varepsilon^2}\left(1 - |u|^2\right)^2\right\} \tag{7}$$

when no external field is applied. Here the unknown is a complex-valued function u. There is a rich literature about the asymptotic behaviour, as $\varepsilon \to 0$, of critical points satisfying a logarithmic energy bound such as (6). It is well-known that, under appropriate assumptions, critical points converge to maps with topology-driven singularities of codimension one. In two-dimensional domains, the theory has been developed after Bethuel, Brezis and Hélein's work [11]. In the three-dimensional case, the asymptotic analysis of minimizers was performed by Lin and Rivière [44], and extended to non-minimizing critical points by Bethuel, Brezis and Orlandi [12]. Later, Jerrard and Soner [40] and Alberti, Baldo, Orlandi [1] proved independently that $|\log\varepsilon|^{-1}E_\varepsilon^{\mathrm{GL}}$ Γ-converges, when $\varepsilon \to 0$, to a functional on integral currents of codimension two. This functional essentially measures the length of defect lines, weighted by some quantity that account for the topology of the defect.

1.3 Main results

For each fixed $\varepsilon > 0$, a classical argument of Calculus of Variations shows that minimizers of (LG$_\varepsilon$) exist as soon as $g_\varepsilon \in H^{1/2}(\partial\Omega,\, \mathbf{S}_0)$ and are regular in the interior of the domain. Our main result deals with their asymptotic behaviour as $\varepsilon \to 0$.

Theorem 1. *Let Ω be a bounded, Lipschitz domain. Assume that there exists a positive constant M such that, for any $0 < \varepsilon < 1$, there hold*

$$E_\varepsilon(Q_\varepsilon) \leq M\left(|\log \varepsilon| + 1\right) \qquad \text{and} \qquad \|Q_\varepsilon\|_{L^\infty(\Omega)} \leq M. \tag{H}$$

Then, there exist a subsequence $\varepsilon_n \searrow 0$, a relatively closed set $\mathscr{S}_{\mathrm{line}} \subseteq \Omega$ and a map $Q_0 \in H^1_{\mathrm{loc}}(\Omega \setminus \mathscr{S}_{\mathrm{line}}, \mathscr{N})$ such that the following holds.

(i) $\mathscr{S}_{\mathrm{line}}$ is a countably $\mathscr{H}^1$-rectifiable set, and $\mathscr{H}^1(\mathscr{S}_{\mathrm{line}}) < +\infty$.

(ii) For any open set $K \subset\subset \Omega$, either $\mathscr{S}_{\mathrm{line}} \cap K$ is empty or it has Hausdorff dimension equal to 1.

(iii) $Q_{\varepsilon_n} \to Q_0$ strongly in $H^1_{\mathrm{loc}}(\Omega \setminus \mathscr{S}_{\mathrm{line}}, \mathscr{N})$.

(iv) Q_0 is locally minimizing harmonic in $\Omega \setminus \mathscr{S}_{\mathrm{line}}$, that is for every ball $B \subset\subset \Omega \setminus \mathscr{S}_{\mathrm{line}}$ and any $P \in H^1(B, \mathscr{N})$, if $P_{|\partial B} = Q_{0|\partial B}$ then

$$\frac{1}{2}\int_B |\nabla Q_0|^2 \leq \frac{1}{2}\int_B |\nabla P|^2 .$$

(v) There exists a locally finite set $\mathscr{S}_{\mathrm{pts}} \subseteq \Omega \setminus \mathscr{S}_{\mathrm{line}}$ such that Q_0 is smooth on $\Omega \setminus (\mathscr{S}_{\mathrm{line}} \cup \mathscr{S}_{\mathrm{pts}})$ and $Q_\varepsilon \to Q_0$ locally uniformly in $\Omega \setminus (\mathscr{S}_{\mathrm{line}} \cup \mathscr{S}_{\mathrm{pts}})$.

By saying that $\mathscr{S}_{\mathrm{line}}$ is countably $\mathscr{H}^1$-rectifiable we mean that there exists a decomposition

$$\mathscr{S}_{\mathrm{line}} = \bigcup_{j\in\mathbb{N}} \mathscr{S}_j,$$

where $\mathscr{H}^1(\mathscr{S}_0) = 0$ and, for each $j \geq 1$, the set $\mathscr{S}_j$ is the image of a Lipschitz function $\mathbb{R} \to \mathbb{R}^3$. Roughly speaking, Condition (ii) means that $\mathscr{S}_{\mathrm{line}}$ contains no such things as Cantor-type components. In addition to the singular set $\mathscr{S}_{\mathrm{line}}$ of dimension one, the limiting map Q_0 may have a set of point singularities $\mathscr{S}_{\mathrm{pts}}$. This is consistent with the regularity results for minimizing harmonic maps [31, 55]. The set $\mathscr{S}_{\mathrm{pts}}$ is locally finite in $\Omega \setminus \mathscr{S}_{\mathrm{line}}$, i.e. for any $K \subset\subset \Omega \setminus \mathscr{S}_{\mathrm{line}}$ the set $\mathscr{S}_{\mathrm{pts}} \cap K$ is finite. Later on, we will discuss examples where $\mathscr{S}_{\mathrm{line}}$ and $\mathscr{S}_{\mathrm{pts}}$ are non-empty.

Theorem 1 is local in nature. In particular, boundary conditions play no particular role in the proof of this result, although they need to be imposed to induce non-trivial behaviour of minimizers.

In Subsection 4.4, we address the analysis near the boundary of the domain, and we prove a weak compactness result for minimizers, under an additional assumption on the boundary datum. As for the properties of the singular set $\mathscr{S}_{\mathrm{line}}$, we also prove the

Proposition 2. *There exists a bounded, $\mathscr{H}^1$-integrable, Borel function $\Theta\colon \mathscr{S}_{\mathrm{line}} \to \mathbb{R}^+$ such that $\mathbf{V}(\mathscr{S}_{\mathrm{line}}, \Theta)$ is a stationary varifold.*

Here $\mathbf{V}(\mathscr{S}_{\mathrm{line}}, \Theta)$ is defined as the equivalence class of all pairs $(\mathscr{S}', \Theta')$, where $\mathscr{S}'$ is a countably $\mathscr{H}^1$-rectifiable set, $\mathscr{H}^1((\mathscr{S}' \setminus \mathscr{S}_{\mathrm{line}}) \cup (\mathscr{S}_{\mathrm{line}} \setminus \mathscr{S}')) = 0$, and $\Theta = \Theta'$ $\mathscr{H}^1$-a.e. on $\mathscr{S}_{\mathrm{line}} \cap \mathscr{S}'$. The definition of stationary varifold is given in [57, Chapter 4]. Varifolds are a generalization of differentiable manifolds, introduced by Almgren [3] in the context of Calulus of Variations. Stationary varifolds can be though as a weak notion of minimal manifolds. Unfortunately, very few regularity results are known for general stationary varifolds. In Proposition 2, both the set $\mathscr{S}_{\mathrm{line}}$ and the density Θ are obtained from the energy density of minimizers Q_ε, passing to the limit as $\varepsilon \to 0$ in a weak sense.

We provide sufficient conditions for the estimate (H) to hold, in terms of the domain and the boundary data. Here is our first condition.

(H_1) Ω is a bounded, smooth domain and $\{g_\varepsilon\}_{0<\varepsilon<1}$ is a bounded family in $H^{1/2}(\partial\Omega,\, \mathscr{N})$.

The uniform $H^{1/2}$-bound is satisfied if, for instance, $g_\varepsilon = g\colon \partial\Omega \to \mathscr{N}$ has a finite number of disclinations. This means, there exists a finite set $\Sigma \subseteq \partial\Omega$ such that g is smooth on $\partial\Omega \setminus \Sigma$ and around each $x_0 \in \Sigma$ the map g can be written as

$$g(\rho,\, \theta) = s_* \left\{ \left(\tau_1 \cos\frac{\theta}{2} + \tau_2 \sin\frac{\theta}{2} \right)^{\otimes 2} - \frac{1}{3}\,\mathrm{Id} \right\} + \text{smooth terms of order } \rho \qquad \text{as } \rho \to 0, \tag{8}$$

where $(\rho,\, \theta)$ are geodesic polar coordinates centered at x_0 and $(\tau_1,\, \tau_2)$ is an orthonormal pair in $\mathbb{R}^3$.

Proposition 3. *Condition* (H_1) *implies* (H).

Alternatively, one can assume

(H_2) $\Omega \subseteq \mathbb{R}^3$ is a bounded Lipschitz domain, and it is bilipschitz equivalent to a handlebody (i.e. a 3-ball with a finite number of handles attached).

(H_3) There exists $M_0 > 0$ such that, for any $0 < \varepsilon < 1$, we have $g_\varepsilon \in (H^1 \cap L^\infty)(\partial\Omega,\, \mathbf{S}_0)$ and

$$E_\varepsilon(g_\varepsilon,\, \partial\Omega) \leq M_0 \left(|\log\varepsilon| + 1\right), \qquad \|g_\varepsilon\|_{L^\infty(\partial\Omega)} \leq M_0.$$

As an example of sequence satisfying (H_3), one can take a sequence of smooth approximations of a map $g \in \partial\Omega \to \mathscr{N}$ of the form (8). For instance, we might take

$$g_\varepsilon(\rho,\, \theta) := \eta_\varepsilon(\rho) g(\rho,\, \theta) \tag{9}$$

where $\eta_\varepsilon \in C^\infty(0,\, +\infty)$ is such that

$$\eta_\varepsilon(0) = \eta_\varepsilon'(0) = 0, \qquad \eta_\varepsilon(\rho) = 1 \text{ if } \rho \geq \varepsilon, \qquad 0 \leq \eta_\varepsilon \leq 1, \qquad |\eta_\varepsilon'| \leq C\varepsilon^{-1}.$$

Proposition 4. *If* (H_2) *and* (H_3) *are satisfied, the* (H) *holds.*

Remark 1. Hypothesis (H_2) is *not* the same as asking Ω to be a bounded Lipschitz domain with connected boundary. Let $K \subseteq \mathbb{S}^3$ be a (open) tubular neighborhood of a trefoil knot. Then K is a solid torus, i.e. K is diffeomorphic to $\mathbb{S}^1 \times B^2_1$, but $\mathbb{S}^3 \setminus K$ is *not* a solid torus. In fact, $\mathbb{S}^3 \setminus K$ is not even a handlebody, because

$$\pi_1(\mathbb{S}^2 \setminus K) = \text{ the knot group of the trefoil knot } = \left\langle x,\, y \,|\, x^2 = y^3 \right\rangle$$

whereas the fundamental group of any handlebody is free. By composing with a stereographic projection, one constructs a smooth domain $\overline{\Omega} \subseteq \mathbb{R}^3$ diffeomorphic to $\mathbb{S}^3 \setminus K$. In particular, $\partial\Omega$ is a torus but Ω does not satisfies (H_2).

We can explicitely give examples where the limit map Q_0 has a line defect. The following result shows that, given *any* bounded smooth domain, one can find a family of boundary data such that the energy of minimizers blows up as $\varepsilon \to 0$.

Proposition 5. *For each bounded domain $\Omega \subseteq \mathbb{R}^3$ of class C^1, there exists a family of boundary data $\{g_\varepsilon\}_{0<\varepsilon<1}$ satisfying* (H_3) *and a number $\alpha > 0$ such that*

$$E_\varepsilon(Q) \geq \alpha \left(|\log\varepsilon| - 1\right)$$

for any $Q \in H^1_{g_\varepsilon}(\Omega,\, \mathbf{S}_0)$ and any $0 < \varepsilon < 1$. In particular, there is no subsequence of minimizers which converge in $H^1(\Omega,\, \mathbf{S}_0)$.

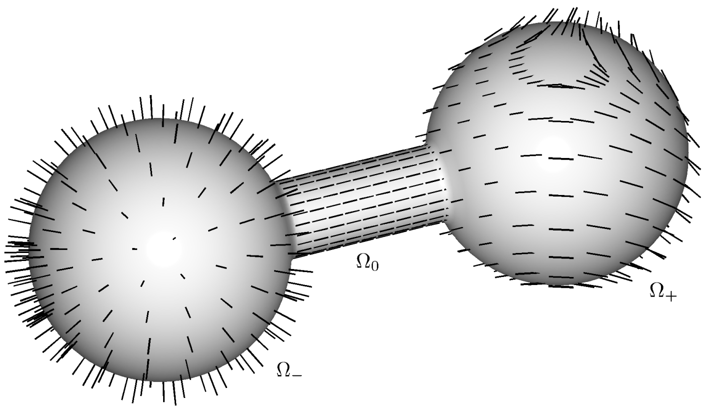


Figure 1: The domain considered in Section 6: two balls (Ω_- on the left, Ω_+ on the right) joined by a cylinder Ω_0 of length $2L$ and radius r. The (unoriented) director field associated to the boundary datum is also represented. The boundary datum restricted to the boundary of Ω_-, Ω_+ defines non-trivial homotopy classes in $\pi_2(\mathscr{N})$, $\pi_1(\mathscr{N})$ respectively.

The functions g_ε are constructed as smooth approximations of a map $\partial\Omega \to \mathscr{N}$ with point singularities of the form (9). There is a topological obstruction to regularity, which is associated with the non-triviality of $\pi_1(\mathscr{N})$, and the set $\mathscr{S}_{\mathrm{line}}$ is non-empty.

Finally, we consider an example where both $\mathscr{S}_{\mathrm{line}}$ and $\mathscr{S}_{\mathrm{pts}}$ are non-empty. The domain consists of two balls, joined by a cylinder of radius $r \in (0, 1/2)$ and length $2L$. The boundary datum, which is defined in Section 6, is uniaxial and has two point defects. The behaviour of the boundary datum is sketched in Figure 1. More precisely, in Figure 1 we represent the direction of the eigenvector corresponding to the leading eigenvalue of the boundary datum (that is, the average orientation of the molecules at each point). This map, restricted to the boundary of the two spheres, defines non-trivial homotopy classes both in $\pi_1(\mathscr{N})$ and in $\pi_2(\mathscr{N})$.

Proposition 6. *There exists a positive number L^* such that, if $L \geq L^*$, then $\mathscr{S}_{\mathrm{line}}$ is non-empty and there exists a point $x_0 \in \mathscr{S}_{\mathrm{pts}}$ such that* $\mathrm{dist}(x_0, \mathscr{S}_{\mathrm{line}}) \geq L/2$.

In other words, if the cylinder is long enough then the limit configuration has both line defects and point defects, which are far away from each other. Although the boundary datum defines a non-trivial class in $\pi_2(\mathscr{N})$, topological arguments alone are not enough to conclude that $\mathscr{S}_{\mathrm{pts}} \neq \emptyset$, for there exist maps $\Omega \to \mathscr{N}$ with $\mathscr{S}_{\mathrm{line}} \neq \emptyset$ but $\mathscr{S}_{\mathrm{pts}} = \emptyset$ (see Remark 6.1). Proposition 6 is inspired by Hardt and Lin's paper [35], where the existence of minimizing harmonic maps with non-topologically induced singularities is proved. However, there is an additional difficulty here, that is minimizers are not uniformly bounded in H^1 as $\varepsilon \to 0$. We take care of this issue by adapting some ideas of the proof of Theorem 1.

Let us spend a few word on the proof of our main result, Theorem 1. The core of the argument is a concentration property for the energy, which can be stated as follows.

Proposition 7. *Assume that the condition* (H) *holds. For any $0 < \theta < 1$ there exist positive numbers η, ϵ_0 and C such that, for any $x_0 \in \Omega$, $R > 0$ satisfying $\overline{B}_R(x_0) \subseteq \Omega$ and any $0 < \varepsilon \leq \epsilon_0\theta R$, if*

$$E_\varepsilon(Q_\varepsilon, B_R(x_0)) \leq \eta R \log \frac{R}{\varepsilon} \tag{10}$$

then

$$E_\varepsilon(Q_\varepsilon, B_{\theta R}(x_0)) \leq CR.$$

Proposition 7 implies that either the energy on a ball blows up at most logarithmically, or it is bounded on a smaller ball. Combining this fact with covering arguments, one proves that the energy concentrates on a set $\mathscr{S}_{\mathrm{line}}$ of finite $\mathscr{H}^1$-measure, and is bounded elsewhere. Then, the asymptotic behaviour of minimizers away from $\mathscr{S}_{\mathrm{line}}$ can be studied using well-established techniques, e.g. arguing as in [47].

Roughly speaking, the proof of Proposition 7 goes as follows. Condition (10) for a small value of η implies that the sphere $\partial B_r(x_0)$ intersect no topological defect line of Q_ε, for a sufficiently large subset of radii $r \in (\theta R, R)$. For in the presence of a topological singularity of codimension two, the energy has order of $\kappa_*|\log \varepsilon|$ for a positive constant κ_*, by Jerrard-Sandier type estimates (see [39, 54]). Because there is no topological obstruction, one can approximate Q_ε with a $\mathscr{N}$-valued map P_ε defined on the sphere $\partial B_r(x_0)$. Then, by adapting Luckhaus' construction [45, Lemma 1], one defines a map φ_ε on a thin spherical shell $B_r(x_0) \setminus B_{r'}(x_0)$, such that $\varphi_\varepsilon = Q_\varepsilon$ on $\partial B_r(x_0)$ and $\varphi_\varepsilon = P_\varepsilon$ on $\partial B_{r'}(x_0)$. Since $\partial B_{r'}(x_0)$ is simply connected, P_ε can be lifted to a $\mathbb{S}^2$-valued map, i.e. one can write

$$P_\varepsilon(x) = s_* \left(\mathbf{n}_\varepsilon^{\otimes 2}(x) - \frac{1}{3} \mathrm{Id} \right) \qquad \text{for } x \in \partial B_{r'}(x_0)$$

for a smooth map $\mathbf{n}_\varepsilon \colon \partial B_{r'}(x_0) \to \mathbb{S}^2$. This is a crucial point in the proof, for it makes possible to apply the methods by Hardt, Kinderlehrer and Lin [34, Lemma 2.3] and obtain boundedness for the energy, by a comparison argument. In other words, on simply connected regions where no obstruction occurs from $\pi_1(\mathscr{N})$, the asymptotic analysis of the Landau-de Gennes problem can be reduced to the analysis of the Oseen-Frank problem, by lifting. Extension results are needed in several steps of this proof, for instance to construct the interpolation map φ_ε. Various results in this direction are discussed in detail in Section 3. In particular, we prove variants of Luckhaus' lemma [45, Lemma 1] which are fit for our purposes.

Remark 2. Theorem 1 implies that the (H) yields compactness for the sequence[2] $\{Q_\varepsilon\}_{0<\varepsilon<1}$. An analogous property does *not* hold for the Ginzburg-Landau energy (7). Indeed, a counter-example by Brezis and Mironescu [18] shows that there exists minimizers $u_\varepsilon \in H^1(B_1^2, \mathbb{C})$ such that

$$E_\varepsilon^{\mathrm{GL}}(u_\varepsilon, B_1^2) \ll |\log \varepsilon| \qquad \text{and} \qquad |u_\varepsilon| \leq 1,$$

yet $\{u_\varepsilon\}_{0<\varepsilon<1}$ does not have subsequences converging a.e. on sets of positive measure. The boundary data $g_\varepsilon := u_{\varepsilon|\partial B_1^2}$ are highly-oscillating $\mathbb{S}^1$-valued maps. In particular, the g_ε's can be lifted to $\mathbb{R}$-valued functions φ_ε (that is $g_\varepsilon = \exp(i\varphi_\varepsilon)$), but (φ_ε) *is not a bounded sequence*. This phenomenon cannot occur in our case, because $\mathscr{N}$-valued maps are lifted to $\mathbb{S}^2$-valued maps, i.e. maps taking valued in a *compact* manifold. Therefore, finiteness of the fundamental group $\pi_1(\mathscr{N})$ yields better compactness properties for minimizers.

1.4 Concluding remarks and open questions

Several questions about the asymptotic behaviour of minimizers of the Landau-de Gennes energy on three-dimensional domains remain open. A first question concerns the behaviour of the singular set $\mathscr{S}_{\mathrm{line}}$. By Theorem 1 and Proposition 4.10, we know that $\mathscr{S}_{\mathrm{line}}$ is 1-rectifiable and a stationary varifold. Unfortunately, this does not implies that $\mathscr{S}_{\mathrm{line}}$ is regular, in the classical sense. However, *if* we knew that the density Θ is integer-valued (up to a multiplicative factor), then Allard's regularity theorem for varifolds [2, Theorem 5.5] would imply that $\mathscr{S}_{\mathrm{line}}$ *is a union of line segments*. By minimality arguments, one is quite naturally led to the same conjecture. Since $\mathscr{S}_{\mathrm{line}}$ is obtained as a limit of a sequence of minimizers, one would expect that it inherits from Q_ε minimizing properties, such as being a set of minimal length. If the domain is convex and the boundary data has a finite number of point singularities $x_1, \ldots, x_p$ of the

[2] Throughout the paper, the word "sequence" will be used to denote family of functions indexed by a continuous parameter as well.

form (8), it is natural to conjecture that $\mathscr{S}_{\mathrm{line}}$ is a union of non-intersecting straight lines connecting the x_i's in pairs. (Notice that, by topological arguments, the number p must be even.)

If additional information on the regularity of $\mathscr{S}_{\mathrm{line}}$ is known (in particular, if $\mathscr{S}_{\mathrm{line}}$ is a union of straight lines), then it would be interesting to study the structure of minimizers Q_ε in the core of line defects. For instance, *does the core of line defects contain biaxial phases?* Contreras and Lamy [23] proved that the core of point singularities, in dimension three, contains biaxial phases when the temperature is low enough, but their proof uses a uniform energy bound such as (5) so it does not apply to singularities of codimension two. However, the analysis of point defects on two-dimensional domains (see e.g. [21, 26]) might indicate that line defects also contain biaxial phases, when the temperature is low. A related issue is the analysis of *singularity profiles*. Let $x_0 \in \mathscr{S}_{\mathrm{line}}$ and let Π be an orthogonal plane to $\mathscr{S}_{\mathrm{line}}$, passing through the point x_0. Set

$$P_{\varepsilon, x_0}(y) := Q_\varepsilon(x_0 + \varepsilon y) \qquad \text{for } y \in \Pi.$$

This defines a bounded sequence in $L^\infty(\Pi, \mathbf{S}_0)$, such that

$$\|\nabla P_{\varepsilon, x_0}\|_{L^2(K)} = \|\nabla Q_\varepsilon\|_{L^2(x_0 + \varepsilon K)} \le C(K) \qquad \text{for every } K \subset\subset \Pi.$$

Therefore, up to a subsequence we have $P_{\varepsilon, x_0} \rightharpoonup P_{x_0}$ in $H^1_{\mathrm{loc}}(\Pi, \mathbf{S}_0)$. The map P_{x_0} contains the information on the fine structure of the defect core. What can be said about P_{x_0}?

In another direction, investigating the asymptotic behaviour of a more general class of functionals in the logarithmic energy regime is a challenging issue. For instance, one may consider functionals with more elastic energy terms and/or choose different potentials, such as the sextic potential

$$f(Q) := -\frac{a_1}{2} \operatorname{tr} Q^2 - \frac{a_2}{3} \operatorname{tr} Q^3 + \frac{a_3}{4} \left(\operatorname{tr} Q^2\right)^2 + \frac{a_4}{5} \left(\operatorname{tr} Q^2\right)\left(\operatorname{tr} Q^3\right) + \frac{a_5}{6} \left(\operatorname{tr} Q^2\right)^3 + \frac{a_5'}{6} \left(\operatorname{tr} Q^3\right)^2$$

(see [24, 33]) or the singular potential proposed by Ball and Majumdar [8]. From this point of view, it is interesting to remark that the proof of Proposition 7 is quite robust, as it is based on variational arguments alone and does not use the structure of the Euler-Lagrange equation. Dealing with the Landau-de Gennes functional in full generality will probably require new techniques, but hopefully the variational arguments presented here could be of help in the study of simple cases.

The paper is organized as follows. Section 2 deals with general facts about the space of Q-tensors and Landau-de Gennes minimizers. In particular, lower estimates for the energy of maps $B^2_1 \to \mathbf{S}_0$ are established in Subsection 2.2, by adapting Jerrard's and Sandier's arguments. Section 3 deals with extension problems. The results of this section are a fundamental tool for the proof of the main results. Section 4 aims at proving Theorem 1. Proposition 7 is proved in Subsection 4.1. The asymptotic analysis away from the singular lines is carried out in Subsection 4.2, whereas the singular set $\mathscr{S}$ is defined and studied in Subsection 4.3. Section 5 deals with the proofs of Propositions 4 and 5. Finally, in Section 6 we construct an explicit example, where the limit configuration Q_0 has both lines and point singularities, and we prove Proposition 6.

2 Preliminary results

Throughout the paper, we will use the following notation. We will denote by $B^k_r(x)$ (or, occasionally, $B^k(x, r)$) the k-dimensional open ball of radius r and center x, and by $\overline{B}^k_r(x)$ the corresponding closed ball. When $k = 3$, we omit the superscript and write $B_r(x)$ instead of $B^3_r(x)$. When $x = 0$, we write B^k_r or B_r. Balls in the matrix space $\mathbf{S}_0$ will be denoted $B^{\mathbf{S}_0}_r(Q)$ or $B^{\mathbf{S}_0}_r$. For any $Q \in H^1(\Omega, \mathbf{S}_0)$ and any k-submanifold $U \subseteq \Omega$, we set

$$e_\varepsilon(Q) := \frac{1}{2} |\nabla Q|^2 + \frac{1}{\varepsilon^2} f(Q), \qquad E_\varepsilon(Q, U) := \int_U e_\varepsilon(Q) \, \mathrm{d}\mathscr{H}^k.$$

The function $e_\varepsilon(Q)$ will be called the energy density of Q. We also set $E_\varepsilon(Q, \emptyset) := 0$ for any map Q. Additional notation will be set later on.

2.1 Properties of $\mathbf{S}_0$ and f

We discuss general facts about Q-tensors, which are useful in order to to have an insight into the structure of the target space $\mathbf{S}_0$. The starting point of our analysis is the following representation formula.

Lemma 2.1. *For all fixed $Q \in \mathbf{S}_0 \setminus \{0\}$, there exist two numbers $s \in (0, +\infty)$, $r \in [0, 1]$ and an orthonormal pair of vectors $(\mathbf{n}, \mathbf{m})$ in $\mathbb{R}^3$ such that*

$$Q = s\left\{\mathbf{n}^{\otimes 2} - \frac{1}{3}\,\mathrm{Id} + r\left(\mathbf{m}^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right)\right\}.$$

Given Q, the parameters $s = s(Q)$, $r = r(Q)$ are uniquely determined. The functions $Q \mapsto s(Q)$ and $Q \mapsto r(Q)$ are continuous on $\mathbf{S}_0 \setminus \{0\}$, and are positively homogeneous of degree 1 and 0, respectively.

Slightly different forms of this formula are often found in the literature (e.g. [47, Proposition 1]). The proof is a straightforward computation sketched in [21, Lemma 3.2], so we omit it here.

Remark 2.1. All the same, we would like to recall some properties of s, r (again, see [21] for a proof). The parameters $s(Q)$, $r(Q)$ are determined by the eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ of Q according to this formula:

$$s(Q) = 2\lambda_1 + \lambda_2, \qquad r(Q) = \frac{\lambda_1 + 2\lambda_2}{2\lambda_1 + \lambda_2}. \tag{2.1}$$

The functions s, r are positively homogeneous of degree one, zero respectively. Following [47, Proposition 15], the vacuum manifold $\mathscr{N} := f^{-1}(0)$ can be characterized as follows:

$$\mathscr{N} = \{Q \in \mathbf{S}_0 \colon s(Q) = s_*,\, r(Q) = 0\},$$

where

$$s_* := \frac{1}{4c}\left(b + \sqrt{b^2 + 24ac}\right).$$

There is another set which is important for our analysis, namely

$$\mathscr{C} := \left\{Q \in \mathbf{S}_0 \setminus \{0\} \colon r(Q) = 1\right\} \cup \{0\}.$$

This is a closed subset of $\mathscr{C}$, and it is cone (i.e., $\lambda Q \in \mathscr{C}$ for any $Q \in \mathscr{C}$, $\lambda \in \mathbb{R}^+$). In view of (2.1), we have

$$\mathscr{C} = \left\{Q \in \mathbf{S}_0 \colon \lambda_1(Q) = \lambda_2(Q)\right\},$$

i.e. $\mathscr{C}$ is the set of matrices whose leading eigenvalue has multiplicity > 1. As a consequence, $Q_0 \in \mathscr{C}$ if and only if the map $Q \mapsto \mathbf{n}(Q)$, where $\mathbf{n}(Q)$ is a unit eigenvector associated with $\lambda_1(Q)$, *fails* to be continuously defined in a neighborhood of Q_0 (see e.g. [7, Section 9.1, Equation (9.1.41), p. 600]). As we will see in a moment, this fact has remarkable consequences on the topological structure of $\mathbf{S}_0$.

Lemma 2.2. *$\mathscr{C} \setminus \{0\}$ is a smooth manifold, diffeomorphic to $\mathbb{RP}^2 \times \mathbb{R}$.*

Proof. We identify $\mathbb{RP}^2$ with the set of matrices $\left\{\mathbf{p}^{\otimes 2} \colon \mathbf{p} \in \mathbb{S}^2\right\} \subseteq \mathrm{M}_3(\mathbb{R})$. Using Lemma 2.1, we can write any $Q \in \mathscr{C} \setminus \{0\}$ in the form

$$Q = s\left(\mathbf{n}^{\otimes 2} + \mathbf{m}^{\otimes 2} - \frac{2}{3}\,\mathrm{Id}\right)$$

for some orthonormal couple of vectors $\mathbf{n}$, $\mathbf{m} \in \mathbb{R}^3$. Set $\mathbf{p} = \mathbf{n} \times \mathbf{m}$, so that $(\mathbf{n}, \mathbf{m}, \mathbf{p})$ is an orthonormal, positively oriented basis in $\mathbb{R}^3$. Using the identity $\mathrm{Id} = \mathbf{n}^{\otimes 2} + \mathbf{m}^{\otimes 2} + \mathbf{p}^{\otimes 2}$, we compute

$$Q = -s\left(\mathbf{p}^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right). \tag{2.2}$$

The eigenvalues of Q, counted with their multiplicity, are $(s/3,\, s/3,\, -2s/3)$, and $\mathbf{p}$ is an eigenvector corresponding to the negative eigenvalue.

In view of (2.2), it is natural to define a map $\varphi\colon \mathscr{C} \setminus \{0\} \to \mathbb{RP}^2 \times (0, +\infty)$ as follows. For a given $Q \in \mathscr{C} \setminus \{0\}$, let $\mathbf{p}$ be a unit eigenvector corresponding to the negative eigenvalue ($\mathbf{p}$ is well-defined up to a sign). Then, set

$$\varphi(Q) := (\mathbf{p}^{\otimes 2},\, s(Q)).$$

This function is well-defined and smooth (because the negative eigenvalue of Q has multiplicity 1, we can apply standard regularity results for the eigenvectors, e.g. [7, Section 9.1 and in particular (9.1.41), p. 600]). The map

$$(\mathbf{p}^{\otimes 2},\, s) \in \mathbb{RP}^2 \times (0, +\infty) \mapsto -s\left(\mathbf{p}^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right) \in \mathscr{C} \setminus \{0\}$$

is also smooth, and is readily checked to be an inverse for φ. Therefore, φ provides the desired homeomorphism. □

The importance of $\mathscr{C}$ is explained by the following lemma.

Lemma 2.3. *The set $\mathbf{S}_0 \setminus \mathscr{C}$ retracts (by deformation) on $\mathscr{N}$.*

Sketch of the proof. With the help of Lemma 2.1, we define a function $H\colon (\mathbf{S}_0 \setminus \mathscr{C}) \times [0,\, 1] \to \mathbf{S}_0$ by

$$H(Q,\, t) := \Big(ts(Q) + (1-t)s_*\Big)\left\{\mathbf{n}^{\otimes 2} - \frac{1}{3}\,\mathrm{Id} + t\, r(Q)\left(\mathbf{m}^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right)\right\}.$$

It is proven in [21, Lemma 3.10] that H is well-defined and continuous. Moreover, for any $Q \in \mathbf{S}_0 \setminus \mathscr{C}$ and $0 \le t \le 1$ we have

$$H(Q,\, 0) \in \mathscr{N}, \qquad H(Q,\, 1) = Q, \qquad H(Q,\, t) = Q \quad \text{if } Q \in \mathscr{N}.$$

Therefore, $H(\cdot,\, 0)$ provides the desired retraction. □

Throughout the paper, we will denote this retraction by $\varrho := H(\cdot,\, 0)\colon \mathbf{S}_0 \setminus \mathscr{C} \to \mathscr{N}$.

Remark 2.2. Given a bounded domain $U \subseteq \mathbb{R}^k$, a non-trivial boundary datum $g \in C^0(\partial U,\, \mathscr{N})$ and a map $Q \in C^0_g(U,\, \mathbf{S}_0)$, Lemma 2.3 implies that $Q^{-1}(\mathscr{C}) \neq \emptyset$. For otherwise $\varrho \circ Q\colon \Omega \to \mathscr{N}$ would be a well-defined, continuous extension of the boundary datum, which is a contradiction. In this sense, the condition $Q \in \mathscr{C}$ identify the regions where topological defects occur.

Lemma 2.4. *The retraction ϱ is of class C^1 on $\mathbf{S}_0 \setminus \mathscr{C}$. Moreover, ϱ coincides with the nearest-point projection onto $\mathscr{N}$, that is*

$$|Q - \varrho(Q)| \le |Q - P| \tag{2.3}$$

holds for any $Q \in \mathbf{S}_0 \setminus \mathscr{C}$ and any $P \in \mathscr{N}$, with strict inequality if $P \neq \varrho(Q)$.

Proof. Fix a matrix $Q \in \mathbf{S}_0 \setminus \mathscr{C}$, and label $\lambda_1(Q) \ge \lambda_2(Q) \ge \lambda_3(Q)$ the eigenvalues of Q. The leading eigenvalue $\lambda_1(Q)$ is simple, because $r(Q) \neq 1$ implies $\lambda_1(Q) \neq \lambda_2(Q)$ by (2.1). Then, classical differentiability results for the eigenvectors (see e.g. [7, Section 9.1]) imply that there exist a C^1 map $\mathbf{n}$, defined on a neighborhood of Q, such that $\mathbf{n}(P)$ is a unit eigenvector associated with the leading eigenvalue $\lambda_1(P)$, for any P close enough to Q. As a consequence, the map

$$\varrho(P) = s_* \left(\mathbf{n}^{\otimes 2}(P) - \frac{1}{3}\,\mathrm{Id}\right)$$

is of class C^1 in a neighborhood of Q.

To show that ϱ is the nearest point projection onto $\mathscr{N}$, we pick an arbitrary $Q \in \mathbf{S}_0 \setminus \mathscr{C}$ and $P \in \mathscr{N}$. By applying Lemma 2.1, we write

$$Q = s\left(\mathbf{n}^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right) + sr\left(\mathbf{m}^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right) \qquad \text{and} \qquad P = s_*\left(\mathbf{p}^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right)$$

for some numbers $s > 0$ and $0 \le r < 1$, some orthonormal pair $(\mathbf{n}, \mathbf{m})$ and some unit vector $\mathbf{p}$. We compute that

$$|Q - P|^2 = \frac{2}{3}s^2\left(r^2 - r + 1\right) + \frac{2}{3}s_* s(1-r) + \frac{2}{3}s_*^2 - 2s_* s\left\{(\mathbf{n}\cdot\mathbf{p})^2 + r(\mathbf{m}\cdot\mathbf{p})^2\right\}.$$

Given s, r, $\mathbf{n}$ and $\mathbf{m}$, we minimize with respect to $\mathbf{p}$ the right-hand side, subject to the constraint

$$(\mathbf{n}\cdot\mathbf{p})^2 + (\mathbf{m}\cdot\mathbf{p})^2 \le 1.$$

One easily see that, since $r < 1$, the minimum is achieved if and only if $\mathbf{p} = \pm\mathbf{n}$, that is $P = \varrho(Q)$. $\square$

We introduce another function, which is involved in the analysis of Subsection 2.2.

Lemma 2.5. *The function $\phi\colon \mathbf{S}_0 \to \mathbb{R}$ given by $\phi(0) = 0$,*

$$\phi(Q) := s_*^{-1} s(Q)(1 - r(Q)) \qquad \text{for } Q \in \mathbf{S}_0 \setminus \{0\}$$

is Lipschitz continuous on $\mathbf{S}_0$, of class C^1 on $\mathbf{S}_0 \setminus \mathscr{C}$ and satisfies

$$\sqrt{2}s_*^{-1} \le |\mathrm{D}\phi(Q)| \le 2s_*^{-1} \qquad \text{for any } Q \in \mathbf{S}_0 \setminus \mathscr{C}.$$

Moreover, $\phi(Q) = 0$ if and only if $Q \in \mathscr{C}$.

Proof. By definition, it is clear that $\phi(Q) = 0$ if and only if $Q = 0$ or $r(Q) = 1$, that is $Q \in \mathscr{C}$. Using (2.1), we can write

$$s_*\phi(Q) = \lambda_1 - \lambda_2, \tag{2.4}$$

where $\lambda_1 \ge \lambda_2 \ge \lambda_3$ are the eigenvalues of Q. Thanks to standard regularity results for the eigenvalues (see e.g. [7, Equation (9.1.32) p. 598]), we immediately deduce that ϕ is locally Lipschitz continuous on $\mathbf{S}_0$ and of class C^1 on $\mathbf{S}_0 \setminus \mathscr{C}$. Let $(\mathbf{n}, \mathbf{m}, \mathbf{p})$ be an orthonormal set of eigenvectors relative to $(\lambda_1, \lambda_2, \lambda_3)$ respectively. Then, for any $Q \in \mathbf{S}_0 \setminus \mathscr{C}$ there holds

$$s_*\left|\mathrm{D}\phi(Q)\right| = \max_{B\in\mathbf{S}_0,\ |B|=1}\left|\frac{\partial\phi}{\partial B}(Q)\right| = \max_{B\in\mathbf{S}_0,\ |B|=1}\left|\mathbf{n}\cdot B\mathbf{n} - \mathbf{m}\cdot B\mathbf{m}\right|$$

(the last identity follows by differentiating (2.4), with the help of [7] again). This implies $|\mathrm{D}\phi(Q)| \le 2$. Now, set

$$B_0 := \frac{1}{\sqrt{2}}\left(\mathbf{n}^{\otimes 2} - \mathbf{m}^{\otimes 2}\right) \in \mathbf{S}_0.$$

Since that $|\mathbf{n}^{\otimes 2}| = |\mathbf{m}^{\otimes 2}| = 1$ and $\mathbf{n}^{\otimes 2}\cdot\mathbf{m}^{\otimes 2} = 0$, it is straightforward to check that $|B_0| = 1$, so

$$s_*\left|\mathrm{D}\phi(Q)\right| \ge \left|\mathbf{n}\cdot B_0\mathbf{n} - \mathbf{m}\cdot B_0\mathbf{m}\right| = \frac{1}{\sqrt{2}}\left(|\mathbf{n}|^2 + |\mathbf{m}|^2\right) = \sqrt{2}. \qquad \square$$

We conclude our discussion on the structure of the target space $\mathbf{S}_0$ by proving a couple of properties of the potential f.

Lemma 2.6. *The Landau-de Gennes potential f, defined by* (3)*, enjoys the following properties. There exists a constant $\gamma_1 = \gamma_1(a, b, c) > 0$ such that*

$$f(Q) \geq \gamma_1 \left(1 - \phi(Q)\right)^2 \qquad \text{for any } Q \in \omega. \tag{F$_1$}$$

Moreover, there exist γ_2, γ_3, $\delta_0 > 0$ such that, if $Q \in \mathbf{S}_0$ satisfies $\mathrm{dist}(Q, \mathscr{N}) \leq \delta_0$*, then*

$$f(Q) \geq \gamma_2 \, \mathrm{dist}^2(Q, \mathscr{N}) \tag{F$_2$}$$

and

$$f\left(tQ + (1-t)\varrho(Q)\right) \leq \gamma_3 t^2 f(Q) \tag{F$_3$}$$

for every $0 \leq t \leq 1$.

Proof of (F$_1$). Using the representation formula of Lemma 2.1, we can compute $\mathrm{tr}\, Q^2$ and $\mathrm{tr}\, Q^3$ as functions of $s := s(Q)$, $t := s(Q)r(Q)$. This yields

$$f(Q) = k - \frac{a}{3}\left(s^2 - st + t^2\right) - \frac{b}{27}\left(2s^3 - 3s^2 t + 3st^2 - 2t^3\right) + \frac{c}{9}\left(s^2 - st + t^2\right)^2 =: \tilde{f}(s, t).$$

We know that $(s_*, 0)$ is the unique minimizer of $\tilde{f}$ (see e.g. [47, Proposition 15]), so $\mathrm{D}^2 \tilde{f}(s_*, 0) \geq 0$. Moreover, it is straightforward to compute that

$$\det \mathrm{D}^2 \tilde{f}(s_*, 0) > 0$$

thus $\mathrm{D}^2 \tilde{f}(s_*, 0) > 0$. As a consequence, there exist two numbers $\delta > 0$ and $C > 0$ such that

$$\tilde{f}(s, sr) \geq C(s_* - s)^2 + Cs^2 r^2 \qquad \text{if } (s - s_*)^2 + s^2 r^2 \leq \delta. \tag{2.5}$$

The left-hand side in this inequality is a polynomial of order four with leading term $\frac{c}{9}(s^2 - st + t^2)^2 \geq \frac{c}{36}(s^2 + t^2)^2$, whereas the right-hand side is a polynomial of order two. Therefore, there exists a positive number M such that

$$\tilde{f}(s, sr) \geq C(s_* - s)^2 + Cs^2 r^2 \qquad \text{if } (s - s_*)^2 + s^2 r^2 \geq M. \tag{2.6}$$

Finally, we have $\tilde{f}(s, t) > 0$ for any $(s, t) \neq (s_*, 0)$, so there exists a positive constant C' such that

$$\tilde{f}(s, sr) \geq C' \qquad \text{if } \delta < (s - s_*)^2 + s^2 r^2 \leq M. \tag{2.7}$$

Combining (2.5), (2.6) and (2.7), and modifying the value of C if necessary, for any $Q \in \mathbf{S}_0$, $s = s(Q)$, $r = r(Q)$ we obtain

$$\tilde{f}(s, sr) \geq C(s_* - s)^2 + Cs^2 r^2 \geq \frac{Cs_*^2}{2}\left(1 - \frac{s}{s_*} + \frac{sr}{s_*}\right)^2 = \frac{Cs_*^2}{2}\left(1 - \phi(Q)\right)^2. \qquad \square$$

Proof of (F$_2$)–(F$_3$). Since the group SO(3) acts transitively on the manifold $\mathscr{N}$ and the potential f is preserved by the action, it suffices to check (F$_2$)–(F$_3$) in a neighborhood of a point $Q_0 \in \mathscr{N}$. Indeed, for any $Q \in \mathbf{S}_0$ there exists $\mathbf{n} \in \mathbb{S}^2$ such that

$$\varrho(Q) = s_* \left(\mathbf{n}\mathbf{n}^\mathsf{T} - \frac{1}{3}\,\mathrm{Id}\right),$$

and there exists a matrix $R \in \mathrm{SO}(3)$ such that $R\mathbf{n} = \mathbf{e}_3$. As is easily checked, the function $\xi_R\colon Q \mapsto RQR^\mathsf{T}$ maps isometrically $\mathbf{S}_0$ onto itself. Then, (2.3) implies that ξ_R commutes with ϱ, so

$$\varrho(\xi_R(Q)) = \xi_R(\varrho(Q)) = s_* \left(R\mathbf{n}(R\mathbf{n})^\mathsf{T} - \frac{1}{3}\,\mathrm{Id}\right) = s_* \left(\mathbf{e}_3\mathbf{e}_3^\mathsf{T} - \frac{1}{3}\,\mathrm{Id}\right) =: Q_0.$$

On the other hand, f is invariant by composition with ξ_R (i.e. $f \circ \xi_R = f$) because it is a function of the scalar invariants of Q. Therefore, if (F$_2$)–(F$_3$) are satisfied in case $\varrho(Q) = Q_0$, then (F$_2$)–(F$_3$) are satisfied for all $Q \in \mathbf{S}_0$ by the same constants γ_2, γ_3, δ_0. Hence, we assume without loss of generality that $\varrho(Q) = Q_0$.

Any matrix $P \in \mathbf{S}_0$ can be written in the form

$$P = \begin{pmatrix} -\frac{1}{3}(s_* + x_0) + x_4 & x_3 & x_1 \\ x_3 & -\frac{1}{3}(s_* + x_0) - x_4 & x_2 \\ x_1 & x_2 & \frac{2}{3}(s_* + x_0) \end{pmatrix}$$

for some $x = (x_0, x_1, \ldots x_4) \in \mathbb{R}^5$. In Lemma [21, Lemma 3.5], it is shown that $P - Q_0 \in T_{Q_0}\mathscr{N}$ if and only if $x_0 = x_3 = x_4 = 0$, and $P - Q_0$ is orthogonal to $T_{Q_0}\mathscr{N}$ if and only if $x_1 = x_2 = 0$. One can write f as a function of x and compute the second derivatives. The computations are straightforward, so we omit them here. One obtains that the hessian matrix $\mathrm{D}^2 f(Q_0)$ is diagonal, with

$$\frac{\partial^2 f}{\partial x_0^2}(Q_0) > 0, \qquad \frac{\partial^2 f}{\partial x_1^2}(Q_0) = \frac{\partial^2 f}{\partial x_2^2}(Q_0) = 0, \qquad \frac{\partial^2 f}{\partial x_3^2}(Q_0) = \frac{\partial^2 f}{\partial x_4^2}(Q_0) > 0.$$

Therefore, there holds

$$0 < \alpha_1 := \min_\nu \frac{1}{2} D^2 f(Q_0)\nu \cdot \nu \leq \alpha_2 := \max_\nu \frac{1}{2} D^2 f(Q_0)\nu \cdot \nu < +\infty,$$

where the minimum and maximum are taken over all $\nu \perp T_{Q_0}\mathscr{N}$ with $|\nu| = 1$. Now, take $P = Q$ with $\varrho(Q) = Q_0$, fix $0 \leq t \leq 1$ and write the Taylor expansion of f around Q_0. The point Q_0 is a minimizer for f, so $\mathrm{D}f(Q_0) = 0$ and

$$f(Q_0 + t(Q - Q_0)) = \frac{t^2}{2}\mathrm{D}^2 f(Q_0)(Q - Q_0) \cdot (Q - Q_0) + \mathrm{o}\left(t^2 (Q - Q_0)^2\right)$$

In particular, if $|Q - Q_0| \leq \delta_0$ and δ_0 is small enough, then

$$\frac{1}{2}\alpha_1 t^2 \left|Q - Q_0\right|^2 \leq f(Q_0 + t(Q - Q_0)) \leq 2\alpha_2 t^2 \left|Q - Q_0\right|^2.$$

The inequality (F$_2$) follows by taking $t = 1$ and setting $\gamma_2 := \alpha_1/2$. As for (F$_3$), combining this upper bound with (F$_2$) we obtain

$$f(Q_0 + t(Q - Q_0)) \leq 2\gamma_2^{-1}\alpha_2 t^2 f(Q),$$

so (F$_3$) is proved for $\gamma_3 := 2\gamma_2^{-1}\alpha_2$. □

2.2 Energy estimates in 2-dimensional domains

In the analysis of the Ginzburg-Landau functional, a very useful tool are the estimates proved by Jerrard [39] and Sandier [54]. These estimates provide a lower bound for the energy of complex-valued maps defined on a two-dimensional disk, depending on the topological properties of the boundary datum. More precisely, if $u \in H^1(B_1^2, \mathbb{C})$ satisfies $|u(x)| = 1$ for a.e. $x \in \partial B_1^2$ (plus some technical assumptions) then

$$E_\varepsilon^{\mathrm{GL}}(u, B_1^2) \geq \pi \, |d| \, |\log \varepsilon| - C, \tag{2.8}$$

where $E_\varepsilon^{\mathrm{GL}}$ is the Ginzburg-Landau energy, defined by (7), and d denotes the topological degree of $u/|u|$, i.e. its winding number. The aim of this subsection is to generalize this result to tensor-valued maps and the Landau-de Gennes energy.

Since we work in the H^1-setting, we have to take care of a technical detail. Set $A := B_1^2 \setminus B_{1/2}^2$. Let $Q \in H^1(B_1^2, \mathbf{S}_0)$ be a given map, which satisfies

$$\phi_0(Q, A) := \operatorname*{ess\,inf}_A \phi \circ Q > 0. \tag{2.9}$$

In case Q is continuous, Condition (2.9) is equivalent to

$$Q(x) \notin \mathscr{C} \qquad \text{for every } x \in \overline{A}.$$

For a.e. $r \in (1/2, 1)$, the restriction of Q to ∂B_r^2 is an H^1-map and hence, by Sobolev injection, a continuous map which satisfies $Q(x) \notin \mathscr{C}$ for every $x \in \partial B_r^2$. Therefore, $\varrho \circ Q$ is well defined and continuous on ∂B_r^2. Moreover, its homotopy class is independent of r. If $\varrho \circ Q$ is continuous, then $\varrho \circ Q$ itself provides a homotopy between $\varrho \circ Q_{|\partial B_{r_1}^2}$ and $\varrho \circ Q_{|\partial B_{r_2}^2}$, for any r_1 and r_2. Otherwise, by convolution (as in [56, Proposition p. 267]) one constructs a regular approximation $(\varrho \circ Q)_\delta \colon A \to \mathscr{N}$ such that $(\varrho \circ Q)_\delta \to \varrho \circ Q$ in $H^1(A, \mathbf{S}_0)$ when $\delta \to 0$. By Sobolev injection, we have $(\varrho \circ Q)_\delta \to \varrho \circ Q$ uniformly on ∂B_r^2 for a.e. $r \in (1/2, 1)$. Therefore, for a.e. r the maps $\varrho \circ Q_{|\partial B_r^2}$ belong to the same homotopy class. By abuse of terminology, this homotopy class will be referred to as "homotopy class of P restricted to the boundary" or also "homotopy class of the boundary datum".

Proposition 2.7. *There exist positive constants M and κ_*, depending only on f, with the following property. Let $0 < \varepsilon < 1$ and $Q \in H^1(B_1^2, \mathbf{S}_0)$ be given. Assume that Q satisfies* (2.9) *and the homotopy class of $\varrho \circ Q_{|\partial B_1^2}$ is non-trivial. Then*

$$E_\varepsilon(Q, B_1^2) \geq \kappa_* \phi_0^2(Q, A) \, |\log \varepsilon| - M.$$

The energetic cost associated with topological defects is quantified by a number κ_*, defined by (2.16) and explicitly computed in Lemma 2.11:

$$\kappa_* = \frac{\pi}{2} s_*^2.$$

This number plays the same role as the quantity $\pi|d|$ in (2.8). The quantity $\phi_0^2(Q, A)$ at the right-hand side has been introduced for technical reasons. Notice that $\phi = 1$ on $\mathscr{N}$, so $\phi_0(Q, A) = 1$ if $Q_{|A}$ take values in $\mathscr{N}$. Before dealing with the proof of Proposition 2.7, we state an immediate consequence.

Corollary 2.8. *Let ε, R be two numbers such that $0 < \varepsilon < R/2$. Let $Q \in H^1(B_R^2, \mathbf{S}_0)$ be such that $Q_{|\partial B_1^2}$ is an H^1-map and*

$$\phi_0(Q, \partial B_R^2) := \operatorname*{ess\,inf}_{\partial B_R^2} \phi \circ Q > 0.$$

If the homotopy class of $\varrho \circ Q_{|\partial B_R^2}$ is non-trivial, then

$$E_\varepsilon(Q, B_R^2) + CR \, E_\varepsilon(Q, \partial B_R^2) \geq \kappa_* \phi_0^2(Q, \partial B_R^2) \log \frac{R}{\varepsilon} - M.$$

If Q satisfies a Dirichlet boundary condition, with boundary datum $g\colon \partial B^2_R \to \mathscr{N}$, then Corollary 2.8 implies the estimate

$$E_\varepsilon(Q,\, B^2_R) \geq \kappa_* \log\frac{R}{\varepsilon} - C,$$

for a constant $M = M(R,\, g)$. (Compare this estimate with [39, Theorem 3.1], [54, Theorem 1], and [22, Proposition 6.1].)

Proof of Corollary 2.8. We apply Proposition 2.7 to $\epsilon := 2\varepsilon/R$ and to the map $\tilde{Q} \in H^1(B^2_1,\, \mathbf{S}_0)$ defined by

$$\tilde{Q}(x) := \begin{cases} Q\left(\dfrac{Rx}{|x|}\right) & \text{if } x \in A := B^2_1 \setminus B^2_{1/2} \\ Q\,(2Rx) & \text{if } x \in B^2_{1/2}. \end{cases}$$

Notice that $\phi_0(\tilde{Q},\, A) = \phi_0(Q,\, \partial B^2_R)$. Then, by a change of variable, we deduce

$$\begin{aligned} \kappa_*\phi^*(Q)\log\frac{R}{\varepsilon} - C \leq E_\epsilon(\tilde{Q},\, B^2_1) &\leq E_\epsilon(Q,\, B^2_{1/2}) + \int_{1/2}^1 E_\epsilon(\tilde{Q},\, \partial B^2_r)\,\mathrm{d}r \\ &= E_\varepsilon(Q,\, B^2_R) + \int_{1/2}^1 \frac{R}{r} E_{2\varepsilon/r}(Q,\, \partial B^2_R)\,\mathrm{d}r \\ &\leq E_\varepsilon(Q,\, B^2_R) + (\log 2) R\, E_\varepsilon(Q,\, \partial B^2_R). \end{aligned}$$

□

A generalization of the Jerrard-Sandier estimate (2.8) has already been proved by Chiron, in his PhD thesis [22]. Given a smooth, compact manifold without boundary, Chiron considered maps into the *cone over* $\mathscr{N}$, that is

$$X_{\mathscr{N}} := ((0,\, +\infty) \times \mathscr{N}) \cup \{0\} \ni u = (|u|,\, u/|u|)$$

(with a metric defined accordingly). He obtained an estimate analogous to (2.8), replacing $\pi|d|$ by some quantity which depends on the homotopy class of u on ∂B^2_1. In case $\mathscr{N} = \mathbb{S}^1$, one has $X_{\mathbb{S}^1} \simeq \mathbb{C}$, so the standard Ginzburg-Landau model is recovered. Given a map $u\colon U \subseteq \mathbb{R}^k \to X_{\mathscr{N}}$, a key step in Chiron's arguments is to decompose the gradient of u in terms of modulus and phase, that is

$$|\nabla u|^2 = |\nabla|u||^2 + |u|^2\,|\nabla\,(u/|u|)|^2 \qquad \text{a.e. on } U. \tag{2.10}$$

Chiron's results do not apply to tensor-valued maps, because the space $\mathbf{S}_0$ do not coincide with the cone over $\mathscr{N}$ (the latter only contains uniaxial matrices, whereas $\mathbf{S}_0$ also contains biaxial matrices). However, one can prove an estimate in the same spirit as (2.10). The energy of a map $\Omega \to \mathbf{S}_0$ is controlled from below by the energy of $\phi\circ Q$ (which plays the role of the modulus) and $\varrho\circ Q$ (in place of the phase).

Lemma 2.9. *Let $U \subseteq \mathbb{R}^k$ be a domain and let $Q \in C^1(U,\, \mathbf{S}_0)$. The function $\varrho\circ Q$ is well-defined and of class C^1 on the open set $Q^{-1}(\mathbf{S}_0 \setminus \mathscr{C}) \subseteq U$, and*

$$|\nabla Q|^2 \geq \frac{s_*^2}{3}\,|\nabla\,(\phi\circ Q)|^2 + (\phi\circ Q)^2\,|\nabla\,(\varrho\circ Q)|^2 \qquad \mathscr{H}^k\text{-a.e. on } U \tag{2.11}$$

(where we have set $(\phi\circ Q)|\nabla(\varrho\circ Q)|(x) := 0$ if $Q(x) \in \mathscr{C}$).

Proof. Because of our choice of the norm, we have

$$|\nabla\psi|^2 = \sum_{i=1}^k |\partial_{x_i}\psi|^2 \tag{2.12}$$

for any scalar or tensor-valued map ψ. Thus, it suffices to prove the inequality where ∇ is replaced by the partial differentiation operator ∂_{x_i}, then sum over $i = 1, \ldots, k$. In view of this remark, without loss of generality we assume that $k = 1$.

Since ϕ is Lipschitz continuous, we know that $\phi\circ Q \in W^{1,\infty}_{\mathrm{loc}}(U)$ and $\phi\circ Q = 0$ on $Q^{-1}(\mathscr{C})$. Therefore, for a.e. $x \in Q^{-1}(\mathscr{C})$ we have $(\phi\circ Q)'(x) = 0$ and (2.11) is trivially satisfied at x. For the rest of the proof, we fix a point $x \in U \setminus Q^{-1}(\mathscr{C})$ so $\phi\circ Q$ is of class C^1 in a neighborhood of x.

Suppose that $r(Q(x)) > 0$. In this case, all the eigenvalues of $Q(x)$ have multiplicity 1. Using Lemma 2.1 and the results in [7], the map Q can be locally written as

$$Q = s\left(\mathbf{n}^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right) + sr\left(\mathbf{m}^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right), \tag{2.13}$$

where s, r, $\mathbf{n}$, $\mathbf{m}$ are C^1 functions defined in a neighborhood of x, satisfying the constraints

$$s > 0, \qquad 0 < r < 1, \qquad |\mathbf{n}| = |\mathbf{m}| = 1, \qquad \mathbf{n}\cdot\mathbf{m} = 0.$$

Then, $\varrho\circ Q$ is of class C^1 in a neighborhood of x, and we can compute $|Q'|$, $|(\varrho\circ Q)'|$ in terms of s, r, $\mathbf{n}$, $\mathbf{m}$ and their derivatives. Setting $t := sr$, a straightforward computation gives

$$s_*^2 (\phi\circ Q)'^2 = s'^2 - 2s't' + t'^2, \qquad |(\varrho\circ Q)'|^2 = 2s_*^2\,|\mathbf{n}'|^2$$

and

$$\begin{aligned}
|Q'|^2 &= \frac{2}{3}\left(s'^2 - s't' + t'^2\right) + 2s^2\,|\mathbf{n}'|^2 + 2t^2\,|\mathbf{m}'|^2 + 4st(\mathbf{n}'\cdot\mathbf{m})(\mathbf{n}\cdot\mathbf{m}') \\
&\geq \frac{s_*^2}{3}(\phi\circ Q)'^2 + 2s^2\left(|\mathbf{n}'|^2 + r^2\,|\mathbf{m}'|^2 + 2r(\mathbf{n}'\cdot\mathbf{m})(\mathbf{n}\cdot\mathbf{m}')\right)
\end{aligned} \tag{2.14}$$

Let $\mathbf{p} := \mathbf{n}\times\mathbf{m}$, so that $(\mathbf{n}, \mathbf{m}, \mathbf{p})$ is an orthonormal, positive definite frame in $\mathbb{R}^3$. By differentiating the orthogonality conditions for $(\mathbf{n}, \mathbf{m}, \mathbf{p})$, we obtain that

$$\mathbf{n}' = \alpha\mathbf{m} + \beta\mathbf{p}, \qquad \mathbf{m}' = -\alpha\mathbf{n} + \gamma\mathbf{p},$$

for some smooth, real-valued functions α, β, γ. Then, from (2.14) and (2.2) we deduce

$$\begin{aligned}
|Q'|^2 - \frac{s_*^2}{3}(\phi\circ Q)'^2 &\geq 2s^2\left(\alpha^2 + \beta^2 + r^2(\alpha^2 + \gamma^2) - 2r\alpha^2\right) \\
&\geq 2s^2(1-r)^2(\alpha^2 + \beta^2) \\
&= s_*^{-2}s^2(1-r)^2\,|(\varrho\circ Q)'|^2 = (\phi\circ Q)^2\,|(\varrho\circ Q)'|^2,
\end{aligned}$$

so (2.11) holds at the point x.

If $r(Q) = 0$ in a neighborhood of x then the function $\mathbf{m}$ might not be well-defined. However, the previous computation still make sense because $t = sr$ vanishes in a neighborhood of x, and from (2.2), (2.14) we deduce that (2.11) holds at x.

We still have to consider a case, namely, $r(Q(x)) = 0$ but $r(Q)$ does not vanish identically in a neighborhood of x. In this case, there exists a sequence $x_k \to x$ such that $r(Q(x_k)) > 0$ for each $k \in \mathbb{N}$. By the previous discussion (2.11) holds at each x_k, and the functions $\phi\circ Q$, P' are continuous (by Lemmas 2.5 and 2.4). Passing to the limit as $k \to +\infty$, we conclude that (2.11) is satisfied at x as well. $\square$

Remark 2.3. Lemma 2.9 holds true, with the same proof, when U is a 1-dimensional manifold. When U is a Riemann manifold of dimension k, the equality (2.12) may not be true but $|\nabla\psi|^2$ is still controlled from below by the sum of $|\partial_{x_i}\psi|^2$. Therefore, we obtain an inequality similar to (2.11), where the right-hand side is multiplied by a constant factor $C \neq 1$. This constant depends on k and on the choice of metric.

The regularity of Q in Lemma 2.9 can be relaxed. We give an independent statement of this fact, since it will be useful later.

Corollary 2.10. *The map $\tau\colon \mathbf{S}_0 \to \mathbf{S}_0$ given by*

$$\tau\colon Q \mapsto \begin{cases} s_*\phi(Q)\varrho(Q) & \text{if } Q \in \mathbf{S}_0 \setminus \mathscr{C} \\ 0 & \text{if } Q \in \mathscr{C} \end{cases}$$

is Lipschitz-continuous. Moreover, for any $Q \in H^1(U, \mathbf{S}_0)$ there holds $\tau \circ Q \in H^1(U, \mathbf{S}_0)$ and

$$\frac{1}{4}\left|\nabla\left(\tau \circ Q\right)\right|^2 \leq \frac{s_*^2}{3}\left|\nabla\left(\phi \circ Q\right)\right|^2 + \left(\phi \circ Q\right)^2\left|\nabla(\varrho \circ Q)\right|^2 \leq \left|\nabla Q\right|^2 \qquad \mathscr{H}^k\text{-a.e. on } U. \tag{2.15}$$

Proof. By differentiating the expression of τ and applying (2.11) to the map $Q = \mathrm{Id}_{\mathbf{S}_0}$, we obtain that

$$\frac{1}{4}\left|\mathrm{D}\tau\right|^2 \leq \frac{s_*^2}{3}\left|\mathrm{D}\phi\right|^2 + \phi^2\left|\mathrm{D}\varrho\right|^2 \leq C \qquad \text{on } \mathbf{S}_0 \setminus \mathscr{C}.$$

Using this uniform bound, together with $\tau \in C(\mathbf{S}_0, \mathbf{S}_0)$ and $\tau_{|\mathscr{C}} = 0$, it is not hard to conclude that τ has bounded derivative in the sense of distributions, therefore τ is a Lipschitz function and the lower bound in (2.15) holds. The upper bound follows easily by a density argument. Let $\{Q^j\}_{j\in\mathbb{N}}$ be a sequence of smooth maps such that $Q^j \to Q$, $\nabla Q^j \to \nabla Q$. Using the regularity of ϱ and ϕ on $\mathbf{S}_0 \setminus \mathscr{C}$ (Lemmas 2.4 and 2.5), we deduce $\nabla(\varrho \circ Q^j) \to \nabla(\varrho \circ Q)$ and $\nabla(\phi \circ Q^j) \to \nabla(\phi \circ Q)$ a.e. on $Q^{-1}(\mathbf{S}_0 \setminus \mathscr{C})$, so (2.15) holds a.e. on $Q^{-1}(\mathbf{S}_0 \setminus \mathscr{C})$. On the other hand, $\nabla(\phi \circ Q) = 0$ a.e. on $Q^{-1}(\mathscr{C}) = (\phi \circ Q)^{-1}(0)$, thus (2.15) holds trivially on $U \setminus Q^{-1}(\mathbf{S}_0 \setminus \mathscr{C})$. □

Following an idea of Chiron [22], we can associate with each homotopy class of maps $\mathbb{S}^1 \to \mathscr{N}$ a positive number which measures the energy cost of that class. In case the underlying manifold is the real projective plane, quantifying the energy cost of homotopically non-trivial maps is simple, because there is a unique homotopy class of such maps. Define

$$\kappa_* := \inf\left\{\frac{1}{2}\int_{\mathbb{S}^1}\left|P'(\theta)\right|^2 \mathrm{d}\theta\colon P \in H^1(\mathbb{S}^1, \mathscr{N}) \text{ is non homotopically trivial}\right\}. \tag{2.16}$$

Thanks to the compact embedding $H^1(\mathbb{S}^1, \mathscr{N}) \hookrightarrow C^0(\mathbb{S}^1, \mathscr{N})$, it is easy to check that the infimum is achieved and minimizers are geodesics in $\mathscr{N}$. Moreover, we have the following property.

Lemma 2.11. *We have*

$$\kappa_* = \frac{\pi}{2}s_*^2$$

and a minimizer for (2.16) is given by

$$P(\theta) := s_*\left(\mathbf{n}_*(\theta)^{\otimes 2} - \frac{1}{3}\mathrm{Id}\right) \qquad \text{for } 0 \leq \theta \leq 2\pi,$$

where $\mathbf{n}_*(\theta) := (\cos(\theta/2), \sin(\theta/2), 0)^{\mathsf{T}}$.

Proof. By considering a suitable auxiliary map, with nice local properties, we will obtain a complete characterization of the geodesics in $\mathscr{N}$. Then, the lemma will follow straightforwardly.

Define the function $\psi\colon \mathbb{S}^2 \to \mathscr{N}$ by

$$\psi(\mathbf{n}) := s_*\left(\mathbf{n}^{\otimes 2} - \frac{1}{3}\mathrm{Id}\right) \qquad \text{for } \mathbf{n} \in \mathbb{S}^2. \tag{2.17}$$

This is the universal covering map of $\mathscr{N} \simeq \mathbb{RP}^2$. Fix $\mathbf{n} \in \mathbb{S}^2$ and a tangent vector $\mathbf{v} \in T_n\mathbb{S}^2$. By differentiating the function $t \mapsto \psi(\mathbf{n} + t\mathbf{v})$, we obtain

$$\langle d\psi(\mathbf{n}),\, \mathbf{v}\rangle = s_* \left(\mathbf{n} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{n}\right),$$

and it follows that

$$|\langle d\psi(\mathbf{n}),\, \mathbf{v}\rangle|^2 = 2s_*^2 \sum_{i,\, j} \left(\mathbf{n}_i\mathbf{v}_j\mathbf{n}_i\mathbf{v}_j + \mathbf{n}_i\mathbf{v}_j\mathbf{v}_i\mathbf{n}_j\right) = 2s_*^2 \left|\mathbf{v}\right|^2 . \tag{2.18}$$

Denote by g, h the first fundamental forms on $\mathbb{S}^2$, $\mathscr{N}$ respectively (that is, the restriction of the euclidean scalar products of $\mathbb{R}^3$, $\mathbf{S}_0$ to the tangent planes of $\mathbb{S}^2 \subseteq \mathbb{R}^3$, $\mathscr{N} \subseteq \mathbf{S}_0$). In terms of pull-back metrics, Equation (2.18) gives

$$\psi^* h = 2s_*^2 g.$$

The scaling factor $2s_*^2$ is constant, so the Levi-Civita connections associated with $\psi^* h$ and g coincide, for the Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right)$$

of the two metrics coincide. As a consequence, a loop P is a geodesic in $\mathscr{N}$ if and only if it can be written as $P = \psi \circ \mathbf{n}$, where $\mathbf{n}\colon [0, 2\pi] \to \mathbb{S}^2$ is a geodesic path in $\mathbb{S}^2$.

Let $\mathbf{n}\colon [0,\, 2\pi] \to \mathbb{S}^2$ be a geodesic path, that is, an arc of great circle. The map $P := \psi \circ \mathbf{n}$ is a loop if and only if $\psi(\mathbf{n}(0)) = \psi(\mathbf{n}(1))$, which means either $\mathbf{n}(0) = \mathbf{n}(1)$ or $\mathbf{n}(0) = -\mathbf{n}(1)$. In the first case, P is homotopically trivial in $\mathscr{N}$. In the second, P is a non-trivial geodesic loop, and its homotopy class generates the fundamental group $\pi_1(\mathscr{N})$. Since there are no other geodesic loops in $\mathscr{N}$, we deduce that any minimizer for (2.16) must be of the form $P = \psi \circ \mathbf{n}$, where $\mathbf{n}$ is half of a great circle in $\mathbb{S}^2$ parametrized by multiples of arc-length. Now the lemma follows from easy computations. □

By adapting Sandier's arguments in [54], we can bound from below the energy of $\mathscr{N}$-valued maps, in terms of the quantity (2.16). We use the following notation: for any $V \subset\subset \mathbb{R}^2$, we define the radius of V as

$$\mathrm{rad}(V) := \inf\left\{\sum_{i=1}^n r_i : \; V \subseteq \bigcup_{i=1}^n B(a_i,\, r_i)\right\}.$$

Lemma 2.12. *Let V be a subdomain of B^2_1 and let $\rho > 0$ be such that $\mathrm{dist}(V,\, \partial B^2_1) \geq 2\rho$. For any $P \in H^1(U \setminus V,\, \mathscr{N})$ such that $P_{|\partial U}$ is homotopically non-trivial, there holds*

$$\frac{1}{2}\int_{U\setminus V} \left|\nabla P\right|^2 \,\mathrm{d}\mathscr{H}^2 \geq \kappa_* \log\frac{\rho}{\mathrm{rad}(V)}.$$

Sketch of the proof. Suppose, at first, that $V = B^2_r$ with $0 < r < 1$ and u is smooth. Then, computing in polar coordinates, we obtain

$$\begin{aligned}\frac{1}{2}\int_{B^2_1\setminus B^2_r} \left|\nabla P\right|^2 \,\mathrm{d}\mathscr{H}^2 &= \frac{1}{2}\int_r^1 \int_{\mathbb{S}^1} \left(\rho\left|\frac{\mathrm{d}P}{\mathrm{d}\rho}\right|^2 + \frac{1}{\rho}\left|\frac{\mathrm{d}P}{\mathrm{d}\theta}\right|^2\right) \mathrm{d}\theta\,\mathrm{d}\rho \\ &\overset{(2.16)}{\geq} \kappa_* \int_r^1 \frac{\mathrm{d}\rho}{\rho} = \kappa_* \log\frac{1}{r}\end{aligned}$$

so the lemma is satisfied for any $0 < \rho \leq 1$. By a density argument, the same estimate holds for any $P \in H^1(B^2_1 \setminus B^2_r,\, \mathscr{N})$. For a general V, the lemma can be proved arguing exactly as in [54, Proposition p. 385]. (Assuming additional $W^{1,\infty}$-bounds on P, the lemma could also be deduced by the arguments of [39, Theorems 3.1 and 4.1].) □

Finally, we can prove the main result of this subsection.

Proof of Proposition 2.7. We argue as in [22, Theorem 6.1] and [21, Proposition 3.11]. As a first step, we suppose that Q is smooth. For the sake of brevity, we set $A := B^2_1 \setminus B^2_{1/2}$ and write ϕ_0 instead of $\phi_0(Q,\, A)$. Remind that $\phi_0 > 0$, by assumption (2.9). There must be

$$\operatorname*{ess\,inf}_{B^2_1} \phi \circ Q = 0, \tag{2.19}$$

otherwise $\varrho \circ Q$ would be a well-defined, continuous map in $H^1(B^2_1,\, \mathscr{N})$ and the boundary datum would be topologically trivial. For each $\lambda > 0$, we set

$$\Omega_\lambda := \left\{x \in B^2_1 \colon \phi \circ Q(x) > \lambda\right\}, \quad \omega_\lambda := \left\{x \in B^2_1 \colon \phi \circ Q(x) < \lambda\right\}, \quad \Gamma_\lambda := \partial\Omega_\lambda \setminus \partial\Omega = \partial\omega_\lambda.$$

Notice that Ω_λ, ω_λ, and Γ_λ are non empty for a.e. $\lambda \in (0,\, \phi_0)$, due to (2.19). We also set

$$\Theta(\lambda) := \int_{\Omega_\lambda} \left|\nabla\left(\varrho \circ Q\right)\right|^2 \mathrm{d}\mathscr{H}^2, \qquad \nu(\lambda) := \int_{\Gamma_\lambda} \left|\nabla\left(\phi \circ Q\right)\right| \mathrm{d}\mathscr{H}^1.$$

Lemma 2.9 entails

$$\int_{B^2_1} |\nabla Q|^2 \geq \int_{B^2_1} \left\{\frac12 \left|\nabla\left(\phi \circ Q\right)\right|^2 + \left(\phi \circ Q\right)^2 \left|\nabla\left(\varrho \circ Q\right)\right|^2\right\} \mathrm{d}\mathscr{H}^2$$

and, applying the coarea formula, we deduce

$$E_\varepsilon(Q) \geq \frac12 \int_0^{\phi_0} \left\{\int_{\Gamma_\lambda} \left(\frac12 \left|\nabla\left(\phi \circ Q\right)\right| + \frac{2f(Q)}{\varepsilon^2 \left|\nabla\left(\phi \circ Q\right)\right|}\right) \mathrm{d}\mathscr{H}^1 - 2\lambda^2\Theta'(\lambda)\right\} \mathrm{d}\lambda\,. \tag{2.20}$$

Thanks to Sard lemma, a.e. $\lambda \in (0,\, \phi_0)$ is a proper regular value of $\phi \circ Q$, so dividing by $|\nabla(\phi \circ Q)|$ makes sense. Let us estimate the terms in the right-hand side of (2.20), starting from the second one. Lemma 2.6, (F_1) implies that

$$f(Q) \geq C\left(1 - \lambda\right)^2 \qquad \mathscr{H}^1\text{-a.e. on } \Gamma_\lambda.$$

Therefore, with the help of the Hölder inequality we deduce

$$\int_{\Gamma_\lambda} \frac{2f(Q)}{\varepsilon^2) \left|\nabla\left(\phi \circ Q\right)\right|} \mathrm{d}\mathscr{H}^1 \geq \frac{C\left(1-\lambda\right)^2}{\varepsilon^2} \int_{\Gamma_\lambda} \frac{1}{\left|\nabla\left(\phi \circ Q\right)\right|} \mathrm{d}\mathscr{H}^1 \geq \frac{C\left(1-\lambda\right)^2 \mathscr{H}^1(\Gamma_\lambda)^2}{\varepsilon^2 \nu(\lambda)}. \tag{2.21}$$

Moreover, we have

$$\mathscr{H}^1(\Gamma_\lambda) \geq 2\mathrm{diam}(\Gamma_\lambda) \geq 4\mathrm{rad}(\omega_\lambda).$$

Combining this with (2.20) and (2.21), we find

$$\begin{aligned} E_\varepsilon(Q) &\geq \frac12 \int_0^{\phi_0} \left\{\frac12 \nu(\lambda) + \frac{C\left(1-\lambda\right)^2 \mathrm{rad}^2(\omega_\lambda)}{\varepsilon^2 \nu(\lambda)}\right\} \mathrm{d}\lambda - \int_0^{\phi_0} \lambda^2\Theta'(\lambda)\,\mathrm{d}\lambda \\ &\geq \int_0^{\phi_0} \frac{C}{\varepsilon} \left|1 - \lambda\right| \mathrm{rad}(\omega_\lambda)\,\mathrm{d}\lambda - \int_0^{\phi_0} \lambda^2\Theta'(\lambda)\,\mathrm{d}\lambda. \end{aligned} \tag{2.22}$$

The second line follows by the elementary inequality $a + b \geq 2\sqrt{ab}$. As for the last term, we integrate by parts. For all $\lambda_0 > 0$, we have

$$-\int_{\lambda_0}^{\phi_0} \lambda^2\Theta'(\lambda)\,\mathrm{d}\lambda = 2\int_{\lambda_0}^{\phi_0} \lambda\Theta(\lambda)\,\mathrm{d}\lambda + {\lambda_0}^2\Theta(\lambda_0) \geq 2\int_{\lambda_0}^{\phi_0} \lambda\Theta(\lambda)\,\mathrm{d}\lambda$$

and, letting $\lambda_0 \to 0$, by monotone convergence ($\Theta \geq 0$, $-\Theta' \geq 0$) we conclude that

$$-\int_0^{\phi_0} \lambda^2 \Theta'(\lambda)\, \mathrm{d}\lambda \geq 2 \int_0^{\phi_0} \lambda \Theta(\lambda)\, \mathrm{d}\lambda.$$

Now, for any $\lambda \in (0,\, \phi_0)$ we have $\omega_\lambda \subseteq B^2_{1/2}$, so $\mathrm{dist}(\omega_\lambda,\, \partial B^2_1) \geq 1/2$. Therefore, by applying Lemma 2.12 we obtain

$$\Theta(\lambda) \geq -\kappa_* \log\left(\mathrm{rad}(\omega_\lambda)\right) - \kappa_* \log 4.$$

Thus, (2.22) implies

$$F_t(Q) \geq \int_0^{\phi_0} \left\{ \frac{C}{\varepsilon} |1-\lambda|\, \mathrm{rad}(\omega_\lambda) - 2\kappa_* \lambda \log\left(\mathrm{rad}(\omega_\lambda)\right) \right\} \mathrm{d}\lambda - C.$$

An easy analysis shows that the function $r \in (0,\, +\infty) \mapsto C\varepsilon^{-1}|1-\lambda| r - 2\kappa_*\lambda \log r$ has a unique minimizer r_*, which is readily computed. As a consequence, we obtain the lower bound

$$\begin{aligned} F_t(Q) &\geq \int_0^{\phi_0} \left\{ 2\kappa_*\lambda - 2\kappa_*\lambda \log \frac{C\varepsilon\kappa_*\lambda}{|1-\lambda|} \right\} \mathrm{d}\lambda - C \\ &= -2\kappa_* \int_0^{\phi_0} \left\{ \lambda \log\varepsilon - \lambda + \lambda \log \frac{C\kappa_*\lambda}{|1-\lambda|} \right\} \mathrm{d}\lambda - C \end{aligned}$$

All the terms are locally integrable functions of λ, so the proposition is proved in case Q is smooth.

Given any Q in H^1, we can reduce to previous case by means of a density argument, inspired by [56, Proposition p. 267]. For $\delta > 0$, let χ^δ be a standard mollification kernel and set $Q^\delta := Q * \chi^\delta$. (In order to define the convolution at the boundary of Ω, we extend Q by standard reflection on a neighborhood of the domain.) Then, $\{Q^\delta\}_{\delta>0}$ is a sequence of smooth maps, which converge to Q strongly in H^1 and, by Sobolev embedding, in L^4. This implies $E_\varepsilon(Q^\delta) \to E_\varepsilon(Q)$ as $\delta \to 0$. Moreover, for any $x \in A$ we have

$$\begin{aligned} \mathrm{dist}\left(\phi\circ Q^\delta(x),\, [\phi_0,\, +\infty)\right) &\leq \fint_{B^2_\delta(x)} \left|\phi\circ Q^\delta(x) - \phi\circ Q(y)\right| \mathrm{d}\mathscr{H}^2(y) \\ &\leq C \fint_{B^2_\delta(x)} \left|Q^\delta(x) - Q(y)\right| \mathrm{d}\mathscr{H}^2(y), \end{aligned} \tag{2.23}$$

where the last inequality follows by the Lipschitz continuity of ϕ (Lemma 2.5). Now, the function $y \mapsto Q^\delta(x) - Q(y)$ is orthogonal to $y \mapsto \chi^\delta(x-y)$ in $L^2(B^2_\delta(x))$. Therefore, we can adapt the Poincaré-Wirtinger inequality to obtain

$$\int_{B^2_\delta(x)} \left|Q^\delta(x) - Q(y)\right| \mathrm{d}\mathscr{H}^2(y) \leq C\delta \int_{B^2_\delta(x)} |\nabla Q|\, \mathrm{d}\mathscr{H}^2 \leq C\delta^2 \left(\int_{B^2_\delta(x)} |\nabla Q|^2\, \mathrm{d}\mathscr{H}^2 \right)^{1/2}.$$

This fact, combined with (2.23), implies

$$\mathrm{dist}\left(\phi\circ Q^\delta(x),\, [\phi_0,\, +\infty)\right) \leq C \left(\int_{B^2_\delta(x)} |\nabla Q|^2\, \mathrm{d}\mathscr{H}^2 \right)^{1/2} \to 0 \qquad \text{uniformly in } x \in A \text{ as } \delta \to 0$$

so, in particular, $\phi_0(Q^\delta,\, A) \to \phi_0(Q,\, A)$ as $\delta \to 0$. Then, since the proposition holds for each Q^δ, by passing to the limit as $\delta \to 0$ we see that it also holds for Q. $\square$

2.3 Basic properties of minimizers

We conclude the preliminary section by recalling recall some basic facts about minimizers of (LG_ε).

Lemma 2.13. *Minimizers Q_ε of* (LG_ε) *exist and are of class C^∞ in the interior of Ω. Moreover, for any $U \subset\subset \Omega$ they satisfy*

$$\varepsilon \left\|\nabla Q_\varepsilon\right\|_{L^\infty(U)} \leq C(U).$$

Sketch of the proof. The existence of minimizers follows by standard method in Calculus of Variations. Minimizers solve the Euler-Lagrange system

$$-\varepsilon^2 \Delta Q_\varepsilon - aQ_\varepsilon - bQ_\varepsilon^2 + \frac{b}{3}\,\mathrm{Id}\left|Q_\varepsilon\right|^2 + c\left|Q_\varepsilon\right|^2 Q_\varepsilon = 0 \tag{2.24}$$

on Ω, in the sense of distributions. The term $\mathrm{Id}\,|Q_\varepsilon^2|$ is a Lagrange multiplier, associated with the tracelessness constraint. The elliptic regularity theory, combined with the uniform L^∞-bound of Assumption (H), implies that each component $Q_{\varepsilon,ij}$ is of class C^∞ in the interior of the domain. The $W^{1,\infty}(U)$-bound follows by interpolation results, see [10, Lemma A.1, A.2]. □

Lemma 2.14 (Monotonicity formula)**.** *Let $x_0 \in \Omega$, and let $0 < r_1 < r_2 < \mathrm{dist}(x_0, \partial\Omega)$. Then*

$$r_1^{-1} E_\varepsilon(Q_\varepsilon, B_{r_1}(x_0)) \leq r_2^{-1} E_\varepsilon(Q_\varepsilon, B_{r_2}(x_0)).$$

The reader is referred to [47, Lemma 2] for a proof.

Lemma 2.15 (Stress-energy identity)**.** *For any $i \in \{1, 2, 3\}$, the minimizers satisfy*

$$\frac{\partial}{\partial x_j}\left(e_\varepsilon(Q_\varepsilon)\delta_{ij} - \frac{\partial Q_\varepsilon}{\partial x_i}\cdot\frac{\partial Q_\varepsilon}{\partial x_j}\right) = 0 \qquad \text{in } \Omega$$

in the sense of distributions.

Proof. Since Q_ε is of class C^∞ in the interior of the domain by Lemma 2.13, we can differentiate the products and use the chain rule. Setting $\partial_i := \partial/\partial x_i$ for the sake of brevity, for each i we have

$$\begin{aligned}
&\partial_j\big(e_\varepsilon(Q_\varepsilon)\delta_{ij} - \partial_i Q_\varepsilon\cdot\partial_j Q_\varepsilon\big)\\
&\qquad = \partial_i\partial_k Q_\varepsilon\cdot\partial_k Q_\varepsilon + \frac{1}{\varepsilon^2}\frac{\partial f(Q_\varepsilon)}{\partial Q_{pq}}\partial_i Q_{\varepsilon,pq} - \partial_i\partial_j Q_\varepsilon\cdot\partial_j Q_\varepsilon - \partial_i Q_\varepsilon\cdot\partial_j\partial_j Q_\varepsilon\\
&\qquad \overset{(2.24)}{=} \partial_k\partial_k Q_\varepsilon\cdot\partial_i Q_\varepsilon - \frac{b}{3}\left|Q_\varepsilon\right|^2\mathrm{Id}\cdot\partial_i Q_\varepsilon - \partial_i Q_\varepsilon\cdot\partial_j\partial_j Q_\varepsilon = 0
\end{aligned}$$

where we have used that $\mathrm{Id}\cdot\partial_i Q_\varepsilon = 0$, because Q_ε is traceless. □

3 Extension properties

3.1 Extension of $\mathbb{S}^2$-valued maps

In some of our arguments, we will encounter extension problems for $\mathscr{N}$-valued maps. This means, given $g\colon \partial B_r^k \to \mathscr{N}$ (for $k \in \mathbb{N}$, $k \geq 2$ and $r > 0$) we look for a map $Q\colon B_r^k \to \mathscr{N}$ satisfying $Q_{|\partial B_r^k} = g$, with a control on the energy of Q. When the datum g is regular enough (say, of class C^1) and satisfies some topological condition, this problem can be reformulated in terms of $\mathbb{S}^2$-valued maps. Indeed, if the homotopy class of g is trivial then g can be *lifted*, i.e. there exists a map $\mathbf{n}\colon \partial B_r^k \to \mathbb{S}^2$, as regular as g, such that the diagram

$$\begin{tikzcd} & \mathbb{S}^2 \arrow[d, "\psi"] \\ \partial B_r^k \arrow[ur, "\mathbf{n}"] \arrow[r, "g"'] & \mathscr{N} \end{tikzcd}$$

commutes. Here ψ is the universal covering map of $\mathscr{N}$, given by (2.17). In other words, the function $\mathbf{n}$ satisfies

$$g(x) = (\psi \circ \mathbf{n})(x) \qquad \text{for (almost) every } x \in \partial B_r^k. \tag{3.1}$$

As $\mathbb{S}^2$ is a simply connected manifold, $\mathbb{S}^2$-valued maps are easier to deal with than $\mathscr{N}$-valued map. Then, by applying again the covering map ψ, we can prove extension results for $\mathscr{N}$-valued maps, which will be crucial in the proof of Proposition 7.

Lemma 3.1. *For any $r > 0$, $k \geq 3$ and any $g \in H^1(\partial B_r^k, \mathscr{N})$, there exists $P \in H^1(B_r^k, \mathscr{N})$ which satisfies $P_{|\partial B_r^k} = g$ and*

$$\int_{B_r^k} |\nabla P|^2 \, \mathrm{d}\mathscr{H}^2 \leq C r^{k/2-1/2} \left(\int_{\partial B_r^k} |\nabla_\top g|^2 \, \mathrm{d}\mathscr{H}^1 \right)^{1/2}$$

for a constant C is independent of g, r.

Lemma 3.2. *There exists a constant $C > 0$ such that, for any $r > 0$ and any $g \in H^1(B_r^2, \mathscr{N})$, there exists $P \in H^1(B_h^2, \mathscr{N})$ satisfying $P_{|\partial B_r^2} = g$ and*

$$\int_{B_r^2} |\nabla P|^2 \, \mathrm{d}\mathscr{H}^2 \leq C r \int_{\partial B_r^2} |\nabla_\top g|^2 \, \mathrm{d}\mathscr{H}^1.$$

In Lemma 3.1, the two sides of the inequality have different homogeneities in v, g. This fact is of main importance, for the arguments of Section 4 rely crucially on it. For the case $k = 2$ (Lemma 3.2), we need to assume that g is defined over the whole of B_r^2, because ∂B_r^2 is not simply connected.

A useful technique to construct extensions of $\mathbb{S}^2$-valued maps has been proposed by Hardt, Kinderlehrer and Lin [34]. Their method combines $\mathbb{R}^3$-valued harmonic extensions with an average argument, in order to find a suitable re-projection $\mathbb{R}^3 \to \mathbb{S}^2$.

Lemma 3.3 (Hardt, Kinderlehrer and Lin, [34])**.** *For all $\mathbf{n} \in H^1(\partial B_r^k, \mathbb{S}^2)$, there exists an extension $\mathbf{w} \in H^1(B_r^k, \mathbb{S}^2)$ which satisfy $\mathbf{w}_{|\partial B_r^k} = \mathbf{n}$,*

$$\int_{B_r^k} |\nabla \mathbf{w}|^2 \, \mathrm{d}\mathscr{H}^k \leq C_k r^{k/2-1/2} \left(\int_{\partial B_r^k} |\nabla_\top \mathbf{n}|^2 \, \mathrm{d}\mathscr{H}^{k-1} \right)^{1/2} \tag{3.2}$$

and

$$\int_{B_r^k} |\nabla \mathbf{w}|^2 \, \mathrm{d}\mathscr{H}^k \leq C_k r \int_{\partial B_r^k} |\nabla_\top \mathbf{n}|^2 \, \mathrm{d}\mathscr{H}^{k-1}. \tag{3.3}$$

Sketch of the proof. The existence of an extension $\mathbf{w}$ which satisfies (3.2) has been proved by Hardt, Kinderleherer and Lin (see [34, proof of Lemma 2.3, Equation (2.3)]). Although the proof has been given in the case $k = 3$, a careful reading shows that the same argument applies word by word to any $k \geq 2$. The same map $\mathbf{w}$ also satisfies (3.3): this follows from [34, proof of Lemma 2.3, second and sixth equation at p. 556]. $\square$

We state now a lifting property for Sobolev maps. This subject has been studied extensively, among others, by Bethuel and Zheng [15], Bourgain, Brezis and Mironescu [16], Bethuel and Chiron [13], Ball and Zarnescu [9] (in particular, in the latter a problem closely related to the Q-tensor theory is considered).

Lemma 3.4. *Let $\mathscr{M}$ be a smooth, simply connected surface (possibly with boundary). Then, any map $g \in H^1(\mathscr{M}, \mathscr{N})$ has a lifting, i.e. there exists $\mathbf{n} \in H^1(\mathscr{M}, \mathbb{S}^2)$ which satisfies (3.1). Moreover,*

$$|\nabla g|^2 = 2s_*^2 |\nabla \mathbf{n}|^2 \qquad \mathscr{H}^2\text{-a.e. on } \mathscr{M}. \tag{3.4}$$

If $\mathscr{M}$ has a boundary then $\mathbf{n}_{|\partial\mathscr{M}}$ is a lifting of $g_{|\partial\mathscr{M}}$, and if $g_{|\partial\mathscr{M}} \in H^1(\partial\mathscr{M}, \mathscr{N})$ then $\mathbf{n}_{|\partial\mathscr{M}} \in H^1(\partial\mathscr{M}, \mathbb{S}^2)$.

Sketch of the proof. The identity (3.4) follows directly by (3.1), by a straightforward computation. The existence of a lifting is a well-known topological fact, when g is of class C^1. In case $g \in H^1$ and $\mathscr{M}$ is a bounded, smooth domain in $\mathbb{R}^2$, the existence of a lifting has been proved by Ball and Zarnescu [9, Theorem 2]. Another possibility is to argue by density of smooth maps in $H^1(\mathscr{M}, \mathscr{N})$ (see [56]). In case $\mathscr{M}$ is a manifold with boundary, one can use a density argument again to construct a lifting with the desidered properties. Actually, every H^1-lifting satisfies to the same regularity properties at the boundary. Indeed, if $\mathbf{n}_1$, $\mathbf{n}_2$ are two H^1-lifting of the same map, then $\mathbf{n}_1 \cdot \mathbf{n}_2$ is an H^1-map $\mathscr{M} \to \{1, -1\}$ and so, by a slicing argument, either $\mathbf{n}_1 = \mathbf{n}_2$ a.e. or $\mathbf{n}_1 = -\mathbf{n}_2$ a.e (see [9, Proposition 2]). □

Combining Lemmas 3.3 and 3.4, we obtain easily the results we need.

Proof of Lemmas 3.1 and 3.2. Consider Lemma 3.1 first. Let $\mathbf{n} \in H^1(\partial B_r^k, \mathbb{S}^2)$ be a lifting of g, whose existence is guaranteed by Lemma 3.4, and let $\mathbf{w} \in H^1(B_r^k, \mathbb{S}^2)$ be the extension given by Lemma 3.3. Then, the map defined by

$$P(x) := s_* \left(\mathbf{w}^{\otimes 2}(x) - \frac{1}{3}\,\mathrm{Id} \right) \qquad \text{for } \mathscr{H}^k\text{-a.e. } x \in B_r^k$$

has the desired properties. The proof of Lemma 3.2 is analogous. □

3.2 Luckhaus' lemma and its variants

When dealing with the asymptotic analysis for minimizers Q_ε of (LG$_\varepsilon$), we will be confronted with the following problem. We assume that $B_1 \subseteq \Omega$, and we aim to compare $E_\varepsilon(Q_\varepsilon, B_1)$ with the energy of a map $P_\varepsilon \colon B_1 \to \mathbf{S}_0$. However, it may be that $P_{\varepsilon|\partial B_1} \neq Q_{\varepsilon|\partial B_1}$, so P_ε is not an admissible comparison map. To correct this, we need to construct a function which interpolates between $P_{\varepsilon|\partial B_1}$ and $Q_{\varepsilon|\partial B_1}$ over an spherical shell.

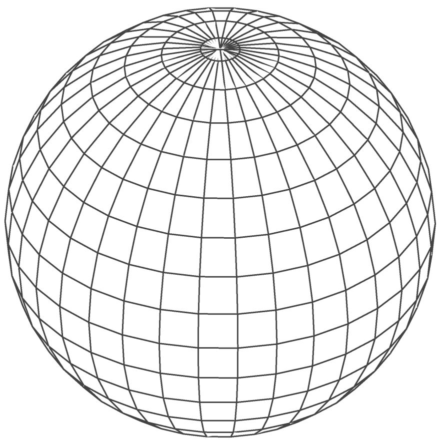
In general terms, the problem may be stated as follows. Fix a parameter $0 < \epsilon < 1$, and consider two H^1-maps u_ϵ, $v_\epsilon \colon \partial B_1 \to \mathbf{S}_0$. We aim at finding a spherical shell $A_\epsilon := B_1 \setminus B_{1-h(\epsilon)}$ of (small) thickness $h(\epsilon) > 0$ and a function $\varphi_\epsilon \colon A_\epsilon \to \mathbf{S}_0$, such that

$$\varphi_\epsilon(x) = u_\epsilon(x) \quad \text{and} \quad \varphi_\epsilon(x - h(\epsilon)x) = v_\epsilon(x) \qquad \text{for } \mathscr{H}^2\text{-a.e. } x \in \partial B_1 \tag{3.5}$$

and the energy $E_\varepsilon(\varphi_\epsilon, A_\epsilon)$ is controlled in terms of u_ϵ, v_ϵ. Additional assumptions on u_ϵ, v_ϵ are needed, otherwise the energy of φ_ϵ may become too large. Moreover, in some circumstances only the function u_ϵ is prescribed, and we will need to find *both* a map $v_\epsilon \colon \partial B_1 \to \mathscr{N}$ which approximates u_ϵ (in some sense to be made precise) and the interpolating function φ_ϵ.

Luckhaus proved an interesting interpolation lemma (see [45, Lemma 1]), which turned out to be useful for several applications. When the two maps u_ϵ, v_ϵ take values in the manifold $\mathscr{N}$, Luckhaus' lemma gives an extension φ_ϵ satisfying (3.5), with bounds on $\mathrm{dist}(\varphi_\epsilon, \mathscr{N})$ and on the Dirichlet integral

$$\int_{B_1 \setminus B_{1-h(\epsilon)}} |\nabla \varphi_\epsilon|^2 .$$



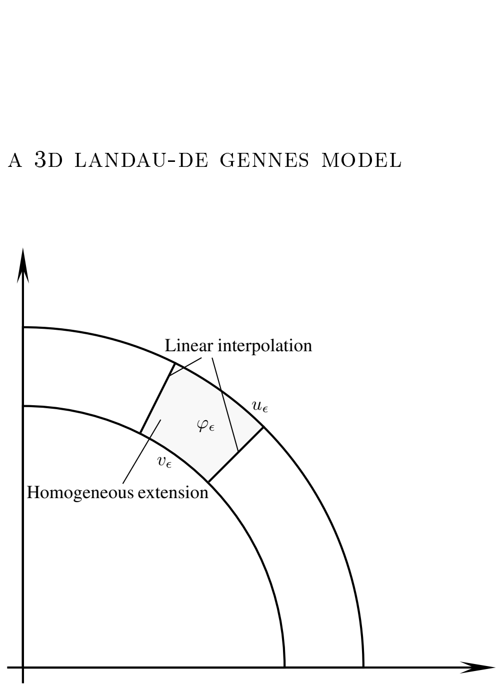


Figure 2: Left: a grid on a sphere. Right: the Luckhaus' construction. Given two maps u_ϵ, v_ϵ (respectively defined on the outer and inner boundary of a thin spherical shell), we construct a map φ_ϵ by using linear interpolation on the boundary of the cells, and homogeneous extension inside each cell.

For the convenience of the reader, and for future reference, we recall Luckhaus' lemma. Since the potential $\epsilon^{-2}f$ is not taken into account here, we drop the subscript ϵ in the notation.

Lemma 3.5 (Luckhaus, [45]). *For any $\beta \in (1/2, 1)$, there exists a constant $C > 0$ with this property. For any fixed numbers $0 < \lambda \leq 1/2$, $0 < \sigma < 1$ and any $u, v \in H^1(\partial B_1, \mathscr{N})$, set*

$$K := \int_{\partial B_1} \left\{ |\nabla u|^2 + |\nabla v|^2 + \frac{|u - v|^2}{\sigma^2} \right\} \mathrm{d}\mathscr{H}^2.$$

Then, there exists a function $\varphi \in H^1(B_1 \setminus B_{1-\lambda}, \mathbf{S}_0)$ satisfying (3.5),

$$\mathrm{dist}(\varphi(x), \mathscr{N}) \leq C\sigma^{1-\beta}\lambda^{-1/2}K^{1/2}$$

for a.e. $x \in B_1 \setminus B_{1-\lambda}$ and

$$\int_{B_1 \setminus B_{1-\lambda}} |\nabla \varphi|^2 \leq C\lambda \left(1 + \sigma^2\lambda^{-2}\right) K.$$

The idea of the proof is illustrated in Figure 2. One constructs a grid on the sphere ∂B_1 with suitable properties. The map φ_ε is defined by linear interpolation between u_ε and v_ε on the boundary of the cells. Inside each cell, φ_ε is defined by a homogeneous extension. By choosing carefully the grid on ∂B_1, and using the Sobolev-Morrey embedding on the boundary of the cells, one can bound the distance between u_ε and v_ε on the 1-skeleton of the grid, in terms of κ. Then, the bound on the energy of φ_ε follows by a simple computation.

We will discuss here a couple of variants of this lemma. In our first result, we suppose that only the map $u_\epsilon \colon \partial B_1 \to \mathbf{S}_0$ is prescribed, so we need to find both $v_\epsilon \colon \partial B_1 \to \mathscr{N}$ and φ_ϵ. Approximating u_ϵ with a $\mathscr{N}$-valued map v_ϵ may be impossible, due to topological obstructions. However, this is possible if the energy of u_ϵ is small, compared to $|\log \epsilon|$. More precisely, we assume that

$$E_\epsilon(u_\epsilon, \partial B_1) \leq \eta_0 \, |\log \epsilon| \tag{3.6}$$

for some small constant $\eta_0 > 0$. For technical reasons, we also require a L^∞-bound on u_ϵ, namely

$$\|u_\epsilon\|_{L^\infty(\partial B_1)} \leq \kappa \tag{3.7}$$

for an ϵ-independent constant κ. In our case of interest, where u_ϵ coincides with a minimizer of $(\mathrm{LG}_\varepsilon)$ restricted on a sphere, (3.7) is guaranteed by (H).

Proposition 3.6. *For any $\kappa > 0$, there exist positive numbers η_0, ϵ_0, C with the following property. For any $0 < \epsilon < \epsilon_0$ and any $u_\epsilon \in (H^1 \cap L^\infty)(\partial B_1, \mathbf{S}_0)$ satisfying (3.6),–(3.7), there exist maps $v_\epsilon \in H^1(\partial B_1, \mathscr{N})$ and $\varphi_\epsilon \in H^1(B_1 \setminus B_{1-h(\epsilon)}, \mathbf{S}_0)$ which satisfy (3.5),*

$$\frac{1}{2}\int_{\partial B_1} |\nabla v_\epsilon|^2 \, \mathrm{d}\mathscr{H}^2 \leq C E_\epsilon(u_\epsilon, \partial B_1), \tag{3.8}$$

$$E_\epsilon(\varphi_\epsilon, B_1 \setminus B_{1-h(\epsilon)}) \leq C h(\epsilon) E_\epsilon(u_\epsilon, \partial B_1) \tag{3.9}$$

for $h(\epsilon) := \epsilon^{1/2}|\log \epsilon|$.

We will discuss the proof of this proposition later on. Before that, we remark that v_ϵ effectively approximates u_ϵ, i.e. their distance — measured in a suitable norm — tends to 0 as $\epsilon \to 0$.

Corollary 3.7. *Under the same assumptions of Proposition 3.6, there holds*

$$\|u_\epsilon - v_\epsilon\|_{L^2(\partial B_1)} \leq C h^{1/2}(\epsilon) E_\epsilon^{1/2}(u_\epsilon, \partial B_1).$$

Proof. We can estimate the L^2-distance between u_n and v_n thanks to (3.5):

$$\begin{aligned} \|u_\epsilon - v_\epsilon\|^2_{L^2(\partial B_1)} &\overset{(3.5)}{=} \int_{\partial B_1} |\varphi_\epsilon(x) - \varphi_\epsilon(x - h(\epsilon)x)|^2 \, \mathrm{d}\mathscr{H}^2(x) \\ &= \int_{\partial B_1} \left| \int_{1-h(\epsilon)}^1 \nabla\varphi_\epsilon(tx)x \, \mathrm{d}t \right|^2 \mathrm{d}\mathscr{H}^2(x). \end{aligned}$$

Then, by Hölder inequality,

$$\begin{aligned} \|u_\epsilon - v_\epsilon\|^2_{L^2(\partial B_1)} &\leq h(\epsilon) \int_{\partial B_1} \int_{1-h(\epsilon)}^1 |\nabla\varphi_\epsilon(tx)|^2 \, \mathrm{d}t \, \mathrm{d}\mathscr{H}^2(x) \\ &\leq \frac{h(\epsilon)}{(1-h(\epsilon))^2} E_\epsilon(\varphi_\epsilon, B_1 \setminus B_{1-h(\epsilon)}) \overset{(3.9)}{\leq} C h(\epsilon) E_\epsilon(u_\epsilon, \partial B_1). \end{aligned}$$ $\square$

Combining Lemma 3.5 and Proposition 3.6, we obtain a third extension result. In this case, both the boundary values u, v are prescribed and, unlike Luckhaus' lemma, we provide a control over the potential energy of the extension $\epsilon^{-2} f(\varphi_\epsilon)$.

Proposition 3.8. *Let $\{\sigma_\epsilon\}_{\epsilon>0}$ be a positive sequence such that $\sigma_\epsilon \to 0$, and let u_ϵ, v_ϵ be given functions in $H^1(\partial B_1, \mathbf{S}_0)$. For all $\epsilon > 0$, assume that u_ϵ satisfies (3.7), that $v_\epsilon(x) \in \mathscr{N}$ for $\mathscr{H}^2$-a.e. $x \in \partial B_1$ and that*

$$\int_{\partial B_1} \left\{ |\nabla u_\epsilon|^2 + \frac{1}{\epsilon^2} f(u_\epsilon) + |\nabla v_\epsilon|^2 + \frac{|u_\epsilon - v_\epsilon|^2}{\sigma_\epsilon^2} \right\} \mathrm{d}\mathscr{H}^2 \leq C \tag{3.10}$$

for an ϵ-independent constant C. Set

$$\nu_\epsilon := h(\epsilon) + \left(h^{1/2}(\epsilon) + \sigma_\epsilon\right)^{1/4} (1 - h(\epsilon)).$$

Then, there exist a number $\epsilon_0 > 0$ and, for $0 < \epsilon \leq \epsilon_0$, a function $\varphi_\epsilon \in H^1(B_1 \setminus B_{1-\nu_\epsilon}, \mathbf{S}_0)$ which satisfies (3.5) and

$$E(\varphi_\epsilon, B_1 \setminus B_{1-\nu_\epsilon}) \leq C\nu_\epsilon.$$

The assumption (3.10) could be relaxed by requiring just a logarithmic bound, of the order of $\eta_0|\log\epsilon|$ for small $\eta_0 > 0$, with additional assumptions on σ_ϵ. However, the result as it is presented here suffices for our purposes.

Proof of Proposition 3.8. Thanks to (3.10) and (3.7), we can apply Proposition 3.6 to the function u_ϵ. We obtain two maps $w_\epsilon \in H^1(\partial B_1, \mathscr{N})$ and $\varphi^1_\epsilon \in H^1(B_1 \setminus B_{1-h(\epsilon)}, \mathbf{S}_0)$, which satisfy

$$\varphi^1_\epsilon(x) = u_\epsilon(x) \quad \text{and} \quad \varphi^1_\epsilon(x - h(\epsilon)x) = w_\epsilon(x) \qquad \text{for } \mathscr{H}^2\text{-a.e. } x \in \partial B_1,$$

$$\int_{\partial B_1} |\nabla w_\epsilon|^2 \, \mathrm{d}\mathscr{H}^2 \leq C,$$

$$E_\epsilon(\varphi^1_\epsilon, B_1 \setminus B_{1-h(\epsilon)}) \leq Ch(\epsilon). \tag{3.11}$$

Corollary 3.7, combined with (3.10), entails

$$\|w_\epsilon - v_\epsilon\|_{L^2(\partial B_1)} \leq \|w_\epsilon - u_\epsilon\|_{L^2(\partial B_1)} + \|u_\epsilon - v_\epsilon\|_{L^2(\partial B_1)} \leq C\left(h^{1/2}(\epsilon) + \sigma_\epsilon\right).$$

Therefore, setting $\tilde\sigma_\epsilon := h^{1/2}(\epsilon) + \sigma_\epsilon$, we have

$$\int_{\partial B_1} \left\{ |\nabla w_\epsilon|^2 + |\nabla v_\epsilon|^2 + \frac{|w_\epsilon - v_\epsilon|^2}{\tilde\sigma^2_\epsilon} \right\} \mathrm{d}\mathscr{H}^2 \leq C$$

Then, we can apply Lemma 3.5 to v_ϵ and w_ϵ, choosing $\sigma = \tilde\sigma_\epsilon$, $\beta = 3/4$ and $\lambda := \tilde\sigma_\epsilon^{1/4}$. By rescaling, we find a map $\varphi^2_\epsilon \in H^1(B_{1-h(\epsilon)} \setminus B_{\nu_\epsilon}, \mathbf{S}_0)$ which satisfies

$$\int_{B_{1-h(\epsilon)} \setminus B_{\nu_\epsilon}} \left|\nabla\varphi^2_\epsilon\right|^2 \leq C\tilde\sigma_\epsilon^{1/4}\,(1 - h(\epsilon))$$

$$\mathrm{dist}(\varphi^2_\epsilon(x), \mathscr{N}) \leq C\tilde\sigma_\epsilon^{1/8} \qquad \text{for all } x \in B_{1-h(\epsilon)} \setminus B_{\nu_\epsilon}. \tag{3.12}$$

Since $\tilde\sigma_\epsilon \to 0$, there exists $\epsilon_0 > 0$ such that $\varphi^2_\epsilon(x) \notin \mathscr{C}$ for any $0 < \epsilon \leq \epsilon_0$ and x. Therefore, the function

$$\varphi_\epsilon(x) := \begin{cases} \varphi^1_\epsilon(x) & \text{if } x \in B_1 \setminus B_{1-h(\epsilon)} \\ \varrho \circ \varphi^2_\epsilon(x) & \text{if } x \in B_{1-h(\epsilon)} \setminus B_{\nu_\epsilon} \end{cases}$$

is well-defined, belongs to $H^1(B_1 \setminus B_{\nu_\epsilon}, \mathscr{N})$, satisfies (3.5) and

$$E_\epsilon(\varphi_\epsilon, B_1 \setminus B_{\nu_\epsilon}) = E_\epsilon(\varphi^1_\epsilon, B_1 \setminus B_{1-h(\epsilon)}) + \int_{B_{1-h(\epsilon)} \setminus B_{\nu_\epsilon}} \left|\nabla\varphi^2_\epsilon\right|^2 \overset{(3.11)\text{–}(3.12)}{\leq} C\nu_\epsilon. \qquad \square$$

Subsections 3.3–3.5 are devoted to the proof of Proposition 3.6, which we sketch here. From now on, we assume that there exists a positive constant M such that

$$E_\epsilon(u_\epsilon, \partial B_1) \leq M\,|\log\epsilon| \qquad \text{for all } 0 < \epsilon < 1. \tag{M$_\epsilon$}$$

As in Luckhaus' arguments, the key ingredient of the construction is the choice of a grid on the unit sphere ∂B_1, with special properties. In Subsection 3.3 we construct a family of grids $\{\mathscr{G}^\epsilon\}$, whose cells have size controlled by $h(\epsilon) = \epsilon^{1/2}|\log\epsilon|$. Assuming that (M$_\epsilon$) holds, we prove that there exists $\epsilon_0 > 0$ such that

$$\mathrm{dist}(u_\epsilon(x), \mathscr{N}) \leq \delta_0 \qquad \text{for any } \epsilon \in (0, \epsilon_0) \text{ and any } x \in R^\epsilon_1.$$

Here R^ϵ_1 denotes the 1-skeleton of $\mathscr{G}^\epsilon$, i.e. the union of all the 1-cells of $\mathscr{G}^\epsilon$, and δ_0 is given by Lemma 2.6. In particular, the composition $\varrho \circ u_\epsilon$ is well-defined on R^ϵ_1 when $\epsilon < \epsilon_0$. It may or may not be possible to extend $\varrho \circ u_{\epsilon|R^\epsilon_1}$ to a map $v_\epsilon \colon \partial B_1 \to \mathscr{N}$ with controlled energy, depending on the homotopy properties of u_ϵ. A sufficient condition for the existence of v_ϵ is the following:

(C_ϵ) For any 2-cell K of $\mathscr{G}^\epsilon$, the loop $\varrho \circ u_{\epsilon|\partial K}\colon \partial K \to \mathscr{N}$ is homotopically trivial.

This condition makes sense for any $u_\epsilon \in H^1(\partial B_1, \mathbf{S}_0)$, for we construct $\mathscr{G}^\epsilon$ in such a way that u_ϵ restricted to the 1-skeleton belongs to $H^1 \hookrightarrow C^0$.

In Subsection 3.4, we assume that (M_ϵ) and (C_ϵ) hold and we construct a function $v_\epsilon \in H^1(\partial B_1, \mathscr{N})$, whose energy is controlled by the energy of u_ϵ. Basically, we extend $\varrho \circ u_{\epsilon|\partial K}$ inside every 2-cell $K \in \mathscr{G}^\epsilon$, which is possible by Condition (C_ϵ). Once v_ϵ is known, we construct φ_ϵ by Luckhaus' method. Particular care must be taken here, as we need to bound the potential energy of φ_ϵ as well.

Finally, in Subsection 3.5 we show that the logarithmic bound (3.6), for a small enough constant η_0, implies that Condition (C_ϵ) is satisfied. Arguing by contra-position, we assume that (C_ϵ) is not satisfied. Then, $\varrho \circ u_{\epsilon|\partial K}$ is non-trivial for at least one 2-cell $K \in \mathscr{G}^\epsilon$. In this case, using Jerrard-Sandier type lower bounds, we prove that the energy $E_\epsilon(u_\epsilon, \partial B_1)$ blows up at least as $\eta_1 |\log \epsilon|$ for some $\eta_1 > 0$. Taking $\eta_0 < \eta_1$, this bound contradicts (3.6) and concludes the proof.

3.3 Good grids on the sphere

Consider a decomposition of ∂B_1 of the form

$$\partial B_1 = \bigcup_{j=0}^{2} \bigcup_{i=1}^{k_j} K_{i,j},$$

where the sets $K_{i,j}$ are mutually disjoint, and each $K_{i,j}$ is bilipschitz equivalent to a j-dimensional ball. The collection of all the $K_{i,j}$'s will be called a *grid* on ∂B_1. Each $K_{i,j}$ will be called a j-cell of the grid. We define the j-skeleton of the grid as

$$R_j := \bigcup_{i=1}^{k_j} K_{i,j} \qquad \text{for } j \in \{0,\, 1,\, 2\}.$$

For our purposes, we need to consider grids with some special properties.

Definition 3.1. Let $h\colon (0,\, \epsilon_0] \to (0,\, +\infty)$ be a fixed function. A *good family of grids* of size h is a family $\mathscr{G} := \{\mathscr{G}^\epsilon\}_{0<\epsilon\leq\epsilon_0}$ of grids on ∂B_1 which satisfies the following properties.

(G_1) There exists a constant $\Lambda > 0$ and, for each ϵ, i, j, a bilipschitz homeomorphism $\varphi^\epsilon_{i,j}\colon K^\epsilon_{i,j} \to B^j_{h(\epsilon)}$ such that
$$\left\| \mathrm{D}\varphi^\epsilon_{i,j} \right\|_{L^\infty} + \left\| \mathrm{D}(\varphi^\epsilon_{i,j})^{-1} \right\|_{L^\infty} \leq \Lambda.$$

(G_2) For all $p \in \{1,\, 2,\, \ldots,\, k_1\}$ we have
$$\left| \left\{ q \in \{1,\, 2,\, \ldots,\, k_2\}\colon K^\epsilon_{p,1} \subseteq K^\epsilon_{q,2} \right\} \right| \leq \Lambda,$$
i.e., each 1-cell is contained in the boundary of at most Λ 2-cells.

(G_3) We have
$$E_\epsilon(u_\epsilon,\, R^\epsilon_1) \leq C h^{-1}(\epsilon)\, E_\epsilon(u_\epsilon,\, \partial B_1),$$
where R^ϵ_1 denotes the 1-skeleton of $\mathscr{G}^\epsilon$.

(G_4) There holds
$$\int_{R^\epsilon_1} f(u_\epsilon)\, \mathrm{d}\mathscr{H}^1 \leq C h^{-1}(\epsilon) \int_{\partial B_1} f(u_\epsilon)\, \mathrm{d}\mathscr{H}^2.$$

Of course, this definition depends on the family $\{u_\epsilon\}$, which we assume to be fixed once and for all.

Lemma 3.9. *For any strictly positive function h, a good family of grids of size h exists.*

Proof. On the unit cube $[0, 1]^3$, consider the uniform grid of size $\lceil h^{-1}(\epsilon)\rceil^{-1}$, i.e. the grid spanned by the points

$$\left(\lceil h^{-1}(\epsilon)\rceil^{-1}\mathbb{Z}^3\right) \cap \partial[0, 1]^3$$

(where $\lceil x\rceil$ is, by the definition, the smallest integer k such that $k \geq x$). By applying a bilipschitz homeomorphism $[0, 1]^3 \to B_1$, one obtains a grid $\mathscr{F}^\epsilon$ on ∂B_1 which satisfy (G_1)–(G_2). Denote by T_1^ϵ the 1-skeleton of $\mathscr{F}^\epsilon$. By an average argument, as in [45, Lemma 1], we find a rotation $\omega \in \mathrm{SO}(3)$ such that

$$E_\epsilon(u_\epsilon, \omega(T_1^\epsilon)) \leq Ch^{-1}(\epsilon)\, E_\epsilon(u_\epsilon, \partial B_1)$$

and

$$\int_{\omega(T_1^\epsilon)} f(u_\epsilon)\, \mathrm{d}\mathscr{H}^1 \leq Ch^{-1}(\epsilon) \int_{\partial B_1} f(u_\epsilon)\, \mathrm{d}\mathscr{H}^2.$$

Thus,

$$\mathscr{G}^\epsilon := \{\omega(K) \colon K \in \mathscr{F}^\epsilon\}$$

is a good family of grids of size h. □

The interest of Definition 3.1 is explained by the following result.

Lemma 3.10. *Let $\mathscr{G}$ be a good family of grids on ∂B_1, of size h. Assume that there exists $\alpha \in (0, 1)$ such that*

$$\lim_{\epsilon\to 0} \epsilon^{-\alpha} h(\epsilon) = +\infty. \tag{3.13}$$

Then, there holds

$$\lim_{\epsilon\to 0} \sup_{x\in R_1^\epsilon} \mathrm{dist}(u_\epsilon(x), \mathscr{N}) = 0.$$

Proof. The arguments below are adapted from [1, Lemmas 3.4 and 3.10] (the reader is also referred to [14, Lemmas 2.2, 2.3 and 2.4]). Since the Landau-de Gennes potential satisfies (F_2) by Lemma 2.6, there exist positive numbers β, C and a continuous function $\psi\colon [0, +\infty) \to \mathbb{R}$ such that

$$\begin{cases} \psi(s) = \beta s^2 & \text{for } 0 \leq s < \delta_0 \\ 0 < \psi(s) \leq C & \text{for } s \geq \delta_0 \\ \psi(\mathrm{dist}(v, \mathscr{N})) \leq f(v) & \text{for any } v \in \mathbf{S}_0. \end{cases}$$

Denote by G a primitive of $\psi^{1/6}$, and set $d_\epsilon := \mathrm{dist}(u_\epsilon, \mathscr{N})$. Since the function $\mathrm{dist}(\cdot, \mathscr{N})$ is 1-Lipschitz continuous, we have $d_\epsilon \in H^1(\Omega, \mathbb{R})$ and $|\nabla d_\epsilon| \leq |\nabla u_\epsilon|$. Moreover, $\psi(d_\epsilon) \leq f(u_\epsilon)$ by construction of ψ. Thus, (M_ϵ) and (G_3) entail

$$C\,|\log\epsilon| \geq h(\epsilon) \int_{R_1^\epsilon} \left\{\frac{1}{2}\,|\nabla d_\epsilon|^2 + \epsilon^{-2}\psi(d_\epsilon)\right\} \mathrm{d}\mathscr{H}^1$$

By applying Young's inequality $a + b \geq Ca^{3/4}b^{1/4}$, we obtain

$$\begin{aligned} C\,|\log\epsilon| &\geq C\epsilon^{-1/2}h(\epsilon) \int_{R_1^\epsilon} |\nabla d_\epsilon|^{3/2}\, \psi^{1/4}(d_\epsilon)\, \mathrm{d}\mathscr{H}^1 \\ &= C\epsilon^{-1/2}h(\epsilon) \int_{R_1^\epsilon} |\nabla G(d_\epsilon)|^{3/2}\, \mathrm{d}\mathscr{H}^1. \end{aligned} \tag{3.14}$$

Fix a 1-cell K of $\mathscr{G}_\epsilon$. With the Sobolev-Morrey embedding $W^{1,3/2}(K) \hookrightarrow C^0(K)$ and (3.14), we can control the oscillations of $G(d_\epsilon)$ over K:

$$\begin{aligned}\left(\operatorname*{osc}_K G(d_\epsilon)\right)^{3/2} &\leq Ch^{1/2}(\epsilon)\int_K |\nabla G(d_\epsilon)|^{3/2} \, \mathrm{d}\mathscr{H}^1 \\ &= C\epsilon^{1/2}h^{-1/2}(\epsilon)\,|\log\epsilon|\,.\end{aligned}$$

A factor $h^{1/2}(\epsilon)$ appears in the right-hand side of this inequality, due to scaling. In view of (3.13), we obtain

$$\operatorname*{osc}_{R_1^\epsilon} G(d_\epsilon) \to 0$$

as $\epsilon \to 0$. But G is a continuous and strictly increasing function, so G has a continuous inverse. This implies

$$\operatorname*{osc}_{R_1^\epsilon} d_\epsilon \to 0 \tag{3.15}$$

as $\epsilon \to 0$. On the other hand, (M_ϵ), (G_3) and (3.13) yield

$$\fint_K \psi(d_\epsilon)\,\mathrm{d}\mathscr{H}^1 \leq \frac{1}{h(\epsilon)}\int_{R_1^\epsilon} f(u_\epsilon)\,\mathrm{d}\mathscr{H}^1 \leq C\epsilon^2 h^{-1}(\epsilon)\,|\log\epsilon| \to 0 \tag{3.16}$$

as $\epsilon \to 0$, for any 1-cell K of $\mathscr{G}_\epsilon$. As we will see in a moment, this implies

$$\sup_K \fint_K d_\epsilon\,\mathrm{d}\mathscr{H}^1 \to 0. \tag{3.17}$$

Therefore, combining (3.17) with (3.15), we conclude that d_ϵ converges uniformly to 0 as $\epsilon \to 0$.

Now, we check that (3.17) holds. There exists a constant $\kappa > 0$ such that

$$\|d_\epsilon\|_{L^\infty(\Omega)} \leq \kappa$$

(this follows from the uniform L^∞-estimate for u_ϵ, (3.7)). For any $\delta \in (0, \kappa)$, set

$$\psi_*(\delta) := \inf_{\delta\leq s\leq\kappa} \psi(s) > 0.$$

Then,

$$\frac{\mathscr{H}^1\left(\{d_\epsilon \geq \delta\}\cap K\right)}{\mathscr{H}^1(K)}\,\psi_*(\delta) \leq \frac{1}{\mathscr{H}^1(K)}\int_{\{d_\epsilon\geq\delta\}\cap K} \psi(d_\epsilon)\,\mathrm{d}\mathscr{H}^1 \leq \fint_K \psi(d_\epsilon)\,\mathrm{d}\mathscr{H}^1. \tag{3.18}$$

Thus, for any 1-cell K, we have

$$\begin{aligned}0 \leq \fint_K d_\epsilon\,\mathrm{d}\mathscr{H}^1 &= \frac{1}{\mathscr{H}^1(K)}\int_{\{d_\epsilon\leq\delta\}\cap K} d_\epsilon\,\mathrm{d}\mathscr{H}^1 + \frac{1}{\mathscr{H}^1(K)}\int_{\{d_\epsilon\geq\delta\}\cap K} d_\epsilon\,\mathrm{d}\mathscr{H}^1 \\ &\leq \frac{\mathscr{H}^1\left(\{d_\epsilon \leq \delta\}\cap K\right)}{\mathscr{H}^1(K)}\,\delta + \frac{\mathscr{H}^1\left(\{d_\epsilon \geq \delta\}\cap K\right)}{\mathscr{H}^1(K)}\,\kappa \\ &\overset{(3.18)}{\leq} \delta + \frac{\kappa}{\psi_*(\delta)}\fint_K \psi(d_\epsilon)\,\mathrm{d}\mathscr{H}^1 \\ &\overset{(3.16)}{\leq} \delta + \frac{C\kappa}{\psi_*(\delta)}\epsilon^2 h^{-1}(\epsilon)\,|\log\epsilon|\,.\end{aligned}$$

We pass to the limit first as $\epsilon \to 0$, then as $\delta \to 0$. Using (3.13), we deduce (3.17). $\square$

3.4 Construction of v_ϵ and φ_ϵ

First, we construct the approximating map $v_\epsilon\colon \partial B_1 \to \mathscr{N}$.

Lemma 3.11. *Assume that* (M_ϵ), (C_ϵ) *hold. There exists a* $\epsilon_0 > 0$ *such that, for any* $0 < \epsilon \le \epsilon_0$, *there exists a map* $v_\epsilon \in H^1(\partial B_1, \mathscr{N})$ *which satisfy* (3.8),

$$v_\epsilon(x) = \varrho(u_\epsilon(x)) \qquad \text{and} \qquad |u_\epsilon(x) - v_\epsilon(x)| \le \delta_0 \tag{3.19}$$

for every $x \in R_1^\epsilon$.

Proof. To construct v_ϵ, we take a family $\mathscr{G} = \{\mathscr{G}^\epsilon\}_{\epsilon>0}$ of grids of size

$$h(\epsilon) := \epsilon^{1/2}\,|\log\epsilon| \tag{3.20}$$

(such a family exists by Lemma 3.9). Condition (3.13) is satisfied for $\alpha = 1/2$, so by Lemma 3.10 there exists $\epsilon_0 > 0$ such that

$$\mathrm{dist}(u_\epsilon(x), \mathscr{N}) \le \delta_0 \qquad \text{for any } \epsilon \in (0, \epsilon_0) \text{ and any } x \in R_1^\epsilon. \tag{3.21}$$

The constant δ_0 is given by Lemma 2.6. In particular, the formula

$$v_\epsilon(x) := \varrho(u_\epsilon(x)) \qquad \text{for all } x \in R_1^\epsilon$$

defines a function $v_\epsilon \in H^1(R_1^\epsilon, \mathbf{S}_0)$ which satisfies (3.19).

To extend v_ϵ inside each 2-cell, we take advantage of Lemma 3.2. Fix a 2-cell K of the grid $\mathscr{G}_\epsilon$. Since we assume that Condition (C_ϵ) holds, $v_{\epsilon|\partial K}$ is homotopically trivial and it can be extended to a map $g_{\varepsilon,K} \in H^1(K, \mathscr{N})$. Therefore, with the help of (G_1) and Lemma 3.2 we find $v_{\epsilon,K} \in H^1(K, \mathscr{N})$ such that $v_{\epsilon,K|\partial K} = v_{\epsilon|\partial K}$ and

$$\int_K |\nabla v_{\epsilon,K}|^2 \,\mathrm{d}\mathscr{H}^2 \le Ch(\epsilon)\int_{\partial K} |\nabla v_\epsilon|^2 \,\mathrm{d}\mathscr{H}^1.$$

Define $v_\epsilon\colon \partial B_1 \to \mathscr{N}$ by setting $v_\epsilon := v_{\epsilon,K}$ on each 2-cell K. This function agrees with $v_{\epsilon|R^1_\epsilon}$ previously defined by (3.19), hence the notation is not ambiguous. Moreover, $v_\epsilon \in H^1(\partial B_1, \mathscr{N})$ and

$$\begin{aligned}\int_{\partial B_1} |\nabla v_\epsilon|^2 \,\mathrm{d}\mathscr{H}^2 &\le \sum_K \int_K |\nabla v_\epsilon|^2 \,\mathrm{d}\mathscr{H}^2 \le Ch(\epsilon)\sum_K \int_{\partial K} |\nabla v_\epsilon|^2 \,\mathrm{d}\mathscr{H}^1 \\ &\overset{(G_2)}{\le} Ch(\epsilon)\int_{R_1^\epsilon} |\nabla v_\epsilon|^2 \,\mathrm{d}\mathscr{H}^1 \overset{(3.19)}{\le} Ch(\epsilon)\int_{R_1^\epsilon} |\nabla u_\epsilon|^2 \,\mathrm{d}\mathscr{H}^1 \\ &\overset{(G_3)}{\le} CE_\epsilon(u_\epsilon, \partial B_1),\end{aligned}$$

where the sum runs over all the 2-cells K of $\mathscr{G}_\epsilon$. Thus v_ϵ satisfies (3.8), so Lemma 3.11 is proved. □

Now, we construct the interpolation map $\varphi_\epsilon\colon \partial B_1 \to \mathbf{S}_0$.

Lemma 3.12. *Assume that the conditions* (M_ϵ), (C_ϵ) *are fulfilled. Then, for any* $0 < \epsilon \le \epsilon_0$ *there exists a map* $\varphi_\epsilon \in H^1(B_1 \setminus B_{1-h(\epsilon)}, \mathbf{S}_0)$ *which satisfies* (3.5) *and* (3.9).

Proof. For the sake of simplicity, set $A_\epsilon := B_1 \setminus B_{1-h(\epsilon)}$. The grid $\mathscr{G}^\epsilon$ on ∂B_1 induces a grid $\hat{\mathscr{G}}^\epsilon$ on A_ϵ, whose cells are

$$\hat{K} := \left\{x \in \mathbb{R}^3\colon 1 - h(\epsilon) \le |x| \le 1, \frac{x}{|x|} \in K\right\} \qquad \text{for each } K \in \mathscr{G}^\epsilon.$$

If K is a cell of dimension j, then $\hat{K}$ has dimension $j+1$. For $j \in \{0, 1, 2\}$, we call $\hat{R}^\epsilon_j$ the union of all the $(j+1)$-cells of $\hat{\mathscr{G}}^\epsilon$.

The function φ_ϵ is constructed as follows. If $x \in \partial B_1 \cup \partial B_{1-h(\epsilon)}$, then $\varphi_\varepsilon(x)$ is determined by (3.5). If $x \in \hat{R}^\epsilon_0 \cup \hat{R}^\epsilon_1$, we define $\varphi_\epsilon(x)$ by linear interpolation:

$$\varphi_\epsilon(x) := \frac{1-|x|}{h(\epsilon)}\, u_\epsilon\left(\frac{x}{|x|}\right) + \frac{h(\epsilon)-1+|x|}{h(\epsilon)}\, v_\epsilon\left(\frac{x}{|x|}\right). \tag{3.22}$$

For any 3-cell $\hat{K}$ of $\mathscr{G}_\epsilon$, we extend homogeneously (of degree 0) the function $\varphi_{\epsilon|\partial\hat{K}}$ on $\hat{K}$. This gives a map $\varphi_\epsilon \in H^1(\hat{K})$, because $\hat{K}$ is a cell of dimension 3. As a result, we obtain a map $\varphi_\epsilon \in H^1(A_\epsilon, \mathbf{S}_0)$ which satisfies (3.5).

To complete the proof of the lemma, we only need to bound the energy of φ_ϵ on A_ϵ. Since φ_ϵ has been obtained by homogeneous extension on cells of size $h(\epsilon)$, we have

$$\begin{aligned} E_\epsilon(\varphi_\epsilon, A_\epsilon) &\overset{(G_1)}{\leq} Ch(\epsilon)\sum_{\hat{K}} E_\epsilon(\varphi_\epsilon, \partial\hat{K}) \\ &\overset{(G_2)}{\leq} Ch(\epsilon)\left\{E_\epsilon(u_\epsilon, \partial B_1) + E_\epsilon(v_\epsilon, \partial B_{1-h(\epsilon)}) + E_\epsilon(\varphi_\epsilon, \hat{R}^\epsilon_1)\right\}, \end{aligned} \tag{3.23}$$

where the sum runs over all the 3-cells $\hat{K}$ of $\hat{\mathscr{G}}^\epsilon$. To conclude the proof, we invoke the following fact.

Lemma 3.13. *We have*

$$E_\epsilon(\varphi_\epsilon, \hat{R}^\epsilon_1) \leq C\left(\epsilon^2 h^{-2}(\epsilon) + 1\right) E_\epsilon(u_\epsilon, \partial B_1).$$

From (3.23) and Lemma 3.13 we get

$$\begin{aligned} E_\epsilon(\varphi_\epsilon, A_\epsilon) &\leq Ch(\epsilon)\left\{\left(\epsilon^2 h^{-2}(\epsilon)+1\right) E_\epsilon(u_\epsilon, \partial B_1) + E_\epsilon(v_\epsilon, \partial B_{1-h(\epsilon)})\right\} \\ &\overset{(3.8)}{\leq} Ch(\epsilon)\left(\epsilon^2 h^{-2}(\epsilon)+1\right) E_\epsilon(u_\epsilon, \partial B_1) \end{aligned}$$

and, thanks to our choice (3.20) of $h(\epsilon)$, we conclude that (3.9) holds, so Lemma 3.12 is proved. □

Proof of Lemma 3.13. We consider first the contribution of the potential energy. Thanks to (F_3), (3.22) and (3.19), we deduce that

$$f(\varphi_\epsilon(x)) \leq C\left(\frac{1-|x|}{h(\epsilon)}\right)^2 f\left(u_\epsilon\left(\frac{x}{|x|}\right)\right) \qquad \text{for } x \in \hat{R}^\epsilon_1.$$

By integration, this gives

$$\int_{\hat{R}^\epsilon_1} f(\varphi_\epsilon)\,\mathrm{d}\mathscr{H}^2 \leq Ch(\epsilon)\int_{R^\epsilon_1} f(u_\epsilon)\,\mathrm{d}\mathscr{H}^2. \tag{3.24}$$

Now, we consider the elastic part of the energy. Using again the definition (3.22) of φ_ϵ on $\hat{R}^\epsilon_1$, we have

$$\int_{\hat{R}^\epsilon_1} |\nabla\varphi_\epsilon|^2\,\mathrm{d}\mathscr{H}^2 \leq Ch^{-1}(\epsilon)\int_{R^\epsilon_1} |u_\epsilon - v_\epsilon|^2\,\mathrm{d}\mathscr{H}^1. \tag{3.25}$$

The condition (F_2) on the Landau-de Gennes potential, together with (3.19), implies

$$\int_{R^\epsilon_1} |u_\epsilon - v_\epsilon|^2\,\mathrm{d}\mathscr{H}^1 \leq C\int_{R^\epsilon_1} f(u_\epsilon)\,\mathrm{d}\mathscr{H}^1. \tag{3.26}$$

Using (3.24), (3.25) and (3.26), we deduce that

$$E_\epsilon(\varphi_\epsilon, \hat{R}_1^\epsilon) \leq C\left(h^{-1}(\epsilon) + \epsilon^{-2}h(\epsilon)\right)\int_{R_1^\epsilon} f(u_\epsilon)\,\mathrm{d}\mathscr{H}^1.$$

Because of Condition (G_4) in the definition of a good grid, we obtain

$$E_\epsilon(\varphi_\epsilon, \hat{R}_1^\epsilon) \leq C\left(h^{-2}(\epsilon) + \epsilon^{-2}\right)\int_{\partial B_1} f(u_\epsilon)\,\mathrm{d}\mathscr{H}^2$$

so the lemma follows easily. □

3.5 Logarithmic bounds for the energy imply (C_ϵ)

The aim of this subsection is to establish the following lemma, and conclude the proof of Proposition 3.6.

Lemma 3.14. *There exists $\eta_1 = \eta_1(\mathscr{N}, \Lambda, M, \epsilon_0)$ such that, if $0 < \epsilon < \epsilon_0$ and u_ϵ satisfies* (M_ϵ), (3.7) *but* not (C_ϵ), *then*

$$E_\epsilon(u_\epsilon, \partial B_1) \geq \eta_1 \left|\log \epsilon\right|.$$

Once Lemma 3.14 is proved, Proposition 3.6 follows in an elementary way.

Proof of Proposition 3.6. Choose $\eta_0 := \eta_1/2$. If u_ϵ satisfies (3.6) with this choice of η_0 and (3.7), then it must satisfy Condition (C_ϵ), otherwise Lemma 3.14 would yield a contradiction. Then, the proposition follows by Lemmas 3.11 and 3.12. □

Proof of Lemma 3.14. By assumption, Condition (C_ϵ) is not satisfied, so there exists a 2-cell $K^* \in \mathscr{G}^\epsilon$ such that $\varrho \circ u_{\epsilon|\partial K^*}$ is non-trivial. By Definition 3.1, there exists a bilipschitz homeomorphism $\varphi\colon K_* \to B_{h(\epsilon)}$ which satisfies (G_1). Therefore, up to composition with φ we can assume that K_* is a 2-dimensional disk, $K_* = B^2_{h(\epsilon)}$. Lemma 3.10 implies that $u_\epsilon(x) \notin \mathscr{C}_0$ for every $x \in \partial K_*$, for $0 < \epsilon \leq \epsilon_0$. Then, by applying Corollary 2.8 we deduce

$$E_\epsilon(u_\epsilon, K_*) + Ch(\epsilon)E_\epsilon(u_\epsilon, \partial K_*) \geq \kappa_* \phi_0^2(u_\epsilon, \partial K_*) \log\frac{h(\epsilon)}{\epsilon} - C$$

Notice that $\phi_0(u_\epsilon, \partial K_*) \geq 1/2$ if δ_0 is small enough, because of (3.21). On the other hand, condition (G_3) yields

$$E_\epsilon(u_\epsilon, K_*) + Ch(\epsilon)E_\epsilon(u_\epsilon, \partial K_*) \leq CE_\epsilon(u_\epsilon, \partial B_1).$$

Due to the previous inequalities and (3.20), we infer

$$E_\epsilon(u_\epsilon, \partial B_1) \geq C\left\{\log\left(\epsilon^{-1/2}\left|\log\epsilon\right|\right) - 1\right\} \geq C\left(\frac{1}{2}\left|\log\epsilon\right| - 1\right)$$

for all $0 < \epsilon \leq \epsilon_0 < 1$, so the lemma follows. □

4 The asymptotic analysis of Landau-de Gennes minimizers

4.1 Concentration of the energy: Proof of Proposition 7

The whole section aims at proving Theorem 1. In this subsection, we prove Proposition 7 by applying the results of Section 3.

Let η_0, ϵ_0 be given by Proposition 3.6. Throughout the section, the same symbol C will be used to denote several different constants, possibly depending on θ and ϵ_0, but not on ε, R. To simplify the notation, from now on we assume that $x_0 = 0$. For a fixed $0 < \varepsilon \leq \epsilon_0\theta R$, define the set

$$D^\varepsilon := \left\{ r \in (\theta R,\, R)\colon E_\varepsilon(Q_\varepsilon,\, \partial B_r) \leq \frac{2\eta}{1-\theta}\log\frac{R}{\varepsilon} \right\}.$$

The elements of D^ε are the "good radii", i.e. $r \in D^\varepsilon$ means that we have a control on the energy on the sphere of radius r. Assume that the condition (10) is satisfied. Then, by an average argument we deduce that

$$\mathscr{H}^1(D^\varepsilon) \geq \frac{(1-\theta)R}{2}. \tag{4.1}$$

For any $r \in D^\varepsilon$ we have

$$E_\varepsilon(Q_\varepsilon,\, \partial B_r) \leq \frac{2\eta}{1-\theta}\left(\log\frac{r}{\varepsilon} - \log\theta\right),$$

since $R \leq \theta^{-1} r$. By choosing η small enough, we can assume that

$$E_\varepsilon(Q_\varepsilon,\, \partial B_r) \leq \eta_0 \log\frac{r}{\varepsilon} \qquad \text{for any } r \in D^\varepsilon \text{ and } 0 < \varepsilon \leq \epsilon_0\theta R. \tag{4.2}$$

In particular, our choice of η depends on ϵ_0, η_0, θ.

Lemma 4.1. *For any $0 < \varepsilon \leq \epsilon_0\theta R$ and any $r \in D^\varepsilon$, there holds*

$$E_\varepsilon(Q_\varepsilon,\, B_r) \leq CR\left(E_\varepsilon^{1/2}(Q_\varepsilon,\, \partial B_r) + 1\right).$$

A similar inequality was obtained by Hardt, Kinderlehrer and Lin in [34, Lemma 2.3, Equation (2.3)], and it played a crucial role in the proof of their energy improvement result.

Proof of Lemma 4.1. To simplify the notations, we get rid of r by means of a scaling argument. Set $\epsilon := \varepsilon/r$, and define the function $u_\epsilon\colon B_1 \to \mathbf{S}_0$ by

$$u_\epsilon(x) := Q_\varepsilon(rx) \qquad \text{for all } x \in B_1.$$

Notice that $\epsilon \leq \epsilon_0$, since $\varepsilon \leq \epsilon_0\theta R$ and $\theta R < r$. The lemma will be proved once we show that

$$E_\epsilon(u_\epsilon,\, B_1) \leq CE_\epsilon^{1/2}(u_\epsilon,\, \partial B_1) + 1 \tag{4.3}$$

(multiplying both sides of (4.3) by $r \leq R$ yields the lemma). Since we have assumed that $r \in D^\varepsilon$ we have, by scaling of (4.2),

$$E_\epsilon(u_\epsilon,\, \partial B_1) \leq \eta_0 \left|\log\epsilon\right|.$$

Moreover, u_ϵ satisfies the L^∞-bound (3.7), due to (H). Therefore, we can apply Proposition 3.6 and find $v_\epsilon \in H^1(\partial B_1,\, \mathscr{N})$, $\varphi_\epsilon \in H^1(A_\epsilon,\, \mathbf{S}_0)$ which satisfy

$$\varphi_\epsilon(x) = u_\epsilon(x) \quad \text{and} \quad \varphi_\epsilon(x - h(\epsilon)x) = v_\epsilon(x) \qquad \text{for } \mathscr{H}^2\text{-a.e. } x \in \partial B_1$$

$$\int_{\partial B_1} |\nabla v_\epsilon|^2 \,\mathrm{d}\mathscr{H}^2 \leq CE_\epsilon(u_\epsilon,\, \partial B_1), \tag{4.4}$$

$$E_\epsilon(\varphi_\epsilon,\, A_\epsilon) \leq Ch(\epsilon)E_\epsilon(u_\epsilon,\, \partial B_1). \tag{4.5}$$

Here $h(\epsilon) := \epsilon^{1/2}|\log\epsilon|$ and $A_\epsilon := B_1 \setminus B_{1-h(\epsilon)}$. By applying Lemma 3.1 to v_ϵ, we find a map $w_\epsilon \in H^1(B_1, \mathscr{N})$ such that $w_{\epsilon|\partial B_1}$ and

$$\int_{B_1} |\nabla w_\epsilon|^2 \leq C\left\{\int_{\partial B_1} |\nabla v_\epsilon|^2 \,\mathrm{d}\mathscr{H}^2\right\}^{1/2} \overset{(4.4)}{\leq} CE_\epsilon^{1/2}(u_\epsilon, \partial B_1). \tag{4.6}$$

Now, define the function $\tilde{w}_\epsilon \colon B_1 \to \mathbf{S}_0$ by

$$\tilde{w}_\epsilon(x) := \begin{cases} \varphi_\epsilon(x) & \text{for } x \in A_\epsilon \\ w_\epsilon\left(\dfrac{x}{1-h(\epsilon)}\right) & \text{for } x \in B_{1-h(\epsilon)}. \end{cases}$$

The energy of $\tilde{w}_\epsilon$ in the spherical shell A_ϵ is controlled by (4.5). Due to our choice of the parameter $h(\epsilon)$, we deduce that

$$E_\epsilon(\tilde{w}_\epsilon, A_\epsilon) \leq 1$$

provided that ϵ_0 is small enough. Combining this with (4.6), we obtain

$$E_\epsilon(\tilde{w}_\epsilon, B_1) \leq CE_\epsilon^{1/2}(u_\epsilon, \partial B_1) + 1.$$

But $\tilde{w}_\epsilon$ is an admissible comparison function for u_ϵ on B_1, because $\tilde{w}_\epsilon = u_\epsilon$ on ∂B_1. Thus, the minimality of u_ϵ implies (4.3). □

Lemma 4.1 can be seen as a non-linear differential inequality for the function $y \colon r \in (\theta R, R) \mapsto E_\varepsilon(Q, B_r)$. The conclusion of the proof of Proposition 7 follows now by a simple ODE argument.

Lemma 4.2. *Let α, β be two positive numbers. Let $y \in W^{1,1}([r_0, r_1], \mathbb{R})$ be a function such that $y' \geq 0$ a.e., and let $D \subseteq (r_0, r_1)$ be a measurable set such that $\mathscr{H}^1(D) \geq (r_1 - r_0)/2$. If the function y satisfies*

$$y(r) \leq \alpha y'(r)^{1/2} + \beta \qquad \textit{for } \mathscr{H}^1\textit{-a.e. } r \in D, \tag{4.7}$$

then there holds

$$y(r_0) \leq \beta + \frac{2\alpha^2}{r_1 - r_0}.$$

Proof. If there exists a point $r_* \in (r_0, r_1)$ such that $y(r_*) \leq \beta$, then $y(r_0) \leq \beta$ (because y is an increasing function) and the lemma is proved. Therefore, we can assume without loss of generality that $y - \beta > 0$ on (r_0, r_1). Then, Equation (4.7) and the monotonicity of y imply

$$\frac{y'(r)}{(y(r) - \beta)^2} \geq \alpha^{-2}\mathbb{1}_D(r) \qquad \text{for a.e. } r \in (r_0, r_1)$$

where $\mathbb{1}_D$ is the characteristic function of D (that is, $\mathbb{1}_D(r) = 1$ if $r \in D$ and $\mathbb{1}_D(r) = 0$ otherwise). By integrating this inequality on $(0, r)$, we deduce

$$\frac{1}{y(r_0) - \beta} - \frac{1}{y(r) - \beta} \geq \alpha^{-2}\mathscr{H}^1\left((r_0, r) \cap D\right) \qquad \text{for any } r \in (r_0, r_1).$$

Since we have assumed that $\mathscr{H}^1(D) \geq (r_1 - r_0)/2$, we obtain

$$\mathscr{H}^1((r_0, r) \cap D) \geq \left(r - \frac{r_0 + r_1}{2}\right)^+ := \max\left\{r - \frac{r_0 + r_1}{2},\, 0\right\}$$

so, via an algebraic manipulation,

$$y(r) \geq \beta + \frac{y(r_0) - \beta}{1 - \alpha^{-2}\left(r - (r_0 + r_1)/2\right)^{+}(y(r_0) - \beta)} \qquad \text{for any } r \in (r_0, r_1).$$

Since y is well-defined (and finite) up to $r = r_1$, there must be

$$1 - \frac{r_1 - r_0}{2\alpha^2}\left(y(r_0) - \beta\right) > 0,$$

whence the lemma follows. □

Conclusion of the proof of Proposition 7. Thanks to Lemma 4.1 and (4.1), we can apply Lemma 4.2 to the function $y(r) := E_\varepsilon(Q_\varepsilon, B_r)$, for $r \in (\theta R, R)$, and the set $D := D^\varepsilon$. This yields

$$E_\varepsilon(Q_\varepsilon, B_{\theta R}) \leq CR,$$

so the proposition is proved. □

4.2 Uniform energy bounds imply convergence to a harmonic map

In this subsection, we suppose that minimizers satisfy

$$E_\varepsilon(Q_\varepsilon, B_R(x_0)) \leq CR \tag{4.8}$$

on a ball $B_r(x_0) \subset\subset \Omega$. In interesting situations, where line defects appear, such an estimate is not valid over the whole of the domain. However, (4.8) is satisfied locally, away from a certain singular set. The main result of this subsection is the following:

Proposition 4.3. *Assume that $\overline{B}_R(x_0) \subseteq \Omega$ and that* (4.8) *is satisfied for some $R, C > 0$. Fix $0 < \theta < 1$. Then, there exist a subsequence $\varepsilon_n \searrow 0$ and a map $Q_0 \in H^1(B_{\theta R}(x_0), \mathscr{N})$ such that*

$$Q_{\varepsilon_n} \to Q_0 \qquad \text{strongly in } H^1(B_{\theta R}(x_0), \mathbf{S}_0).$$

The map Q_0 is minimizing harmonic on $B_{\theta R}(x_0)$, that is, for any $Q \in H^1(B_{\theta R}(x_0), \mathscr{N})$ such that $Q = Q_0$ on $\partial B_{\theta R}(x_0)$ there holds

$$\frac{1}{2}\int_{B_{\theta R}(x_0)} |\nabla Q_0|^2 \leq \frac{1}{2}\int_{B_{\theta R}(x_0)} |\nabla Q|^2 .$$

In general, we cannot expect the map Q_0 to be smooth (see the example of Section 6). In contrast, by Schoen and Uhlenbeck's partial regularity result [55, Theorem II] we know that there exists a finite set $\mathscr{S}_{\text{pts}} \subseteq B_{\theta R}(x_0)$ such that Q_0 is smooth on $B_{\theta R}(x_0) \setminus \mathscr{S}_{\text{pts}}$. Accordingly, the sequence $\{Q_{\varepsilon_n}\}$ will not converge uniformly to Q_0 on the whole of $B_{\theta R}(x_0)$, in general, but we can prove the uniform convergence away from the singularities of Q_0.

Proposition 4.4. *Let $K \subseteq B_{\theta R}(x_0)$ be such that Q_0 is smooth on the closure of K. Then $Q_{\varepsilon_n} \to Q_0$ uniformly on K.*

The asymptotic behaviour of minimizers of the Landau-de Gennes functional, in the bounded-energy regime (4.8), was already studied by Majumdar and Zarnescu in [47]. In that paper, H^1-convergence to a harmonic map and local uniform convergence away from the singularities of Q_0 were already proven. However, in our case some extra care must be taken, because of the local nature of our assumption (4.8).

Proof of Proposition 4.3. Up to a translation, we assume that $x_0 = 0$. In view of (4.8), there exists a subsequence $\varepsilon_n \searrow 0$ and a map $Q_0 \in H^1(B_R, \mathbf{S}_0)$ such that

$$Q_{\varepsilon_n} \to Q_0 \qquad \text{weakly in } H^1(B_R, \mathbf{S}_0), \text{ strongly in } L^2(B_R, \mathbf{S}_0) \text{ and a.e.}$$

Using Fatou's lemma and (4.8) again, we also see that

$$\int_{B_R} f(Q_0) \leq \liminf_{n\to+\infty} \varepsilon_n^2 E_{\varepsilon_n}(Q_{\varepsilon_n}, B_R) \leq \liminf_{n\to+\infty} \varepsilon_n^2 CR = 0,$$

hence $f(Q_0) = 0$ a.e. or, equivalently,

$$Q_0(x) \in \mathscr{N} \qquad \text{for a.e. } x \in B_1.$$

By means of a comparison argument, we will prove that Q_{ε_n}'s actually converge *strongly* in H^1. Fatou's lemma combined with (4.8) gives

$$\int_{\theta R}^{R} \liminf_{n\to+\infty} E_{\varepsilon_n}(Q_{\varepsilon_n}, \partial B_r)\, \mathrm{d}r \leq \liminf_{n\to+\infty} E_{\varepsilon_n}(Q_{\varepsilon_n}, B_R \setminus B_{\theta R}) \leq CR. \tag{4.9}$$

Therefore, the set

$$\left\{ r \in (0, R] \colon \liminf_{n\to+\infty} E_{\varepsilon_n}(Q_{\varepsilon_n}, \partial B_r) > \frac{2C}{1-\theta} \right\}$$

must have length $\leq (1-\theta)R/2$, otherwise (4.9) would be violated. In particular, there exist a radius $r \in (\theta R, R]$ and a relabeled subsequence such that

$$E_{\varepsilon_n}(Q_{\varepsilon_n}, \partial B_r) \leq \frac{2C}{1-\theta}.$$

For ease of notation we scale the variables, setting $\epsilon_n := \varepsilon_n / r$,

$$u_n(x) := Q_{\varepsilon_n}(rx) \qquad \text{and} \qquad u_*(x) := Q_0(rx) \qquad \text{for } x \in B_1.$$

The scaled maps satisfy

$$u_n \to u_* \qquad \text{weakly in } H^1(B_1, \mathbf{S}_0), \text{ strongly in } L^2(B_1, \mathbf{S}_0) \text{ and a.e.,} \tag{4.10}$$

$$u_*(x) \in \mathscr{N} \qquad \text{for a.e. } x \in B_1, \tag{4.11}$$

$$E_{\epsilon_n}(u_n, \partial B_1) \leq C. \tag{4.12}$$

By (4.10) and the trace theorem, $u_n \rightharpoonup u_*$ weakly in $H^{1/2}(\partial B_1, \mathbf{S}_0)$ and hence, by compact embedding, strongly in $L^2(\partial B_1, \mathbf{S}_0)$. Moreover, by (4.12) $u_n \rightharpoonup u_*$ weakly in $H^1(\partial B_1, \mathbf{S}_0)$, so

$$\frac{1}{2}\int_{\partial B_1} |\nabla u_*|^2 \, \mathrm{d}\mathscr{H}^2 \leq \limsup_{n\to+\infty} E_{\epsilon_n}(u_n, \partial B_r) \leq C. \tag{4.13}$$

We are going to apply Proposition 3.8 to interpolate between u_n and u_*. Set $\sigma_n := \|u_n - u_*\|_{L^2(\partial B_1)}$. Then $\sigma_n \to 0$ and

$$\int_{\partial B_1} \left\{ |\nabla u_n|^2 + \frac{1}{\epsilon_n} f(u_n) + |\nabla u_*|^2 + \frac{|u_n - u_*|^2}{\sigma_n} \right\} \mathrm{d}\mathscr{H}^2 \leq C,$$

because of (4.12), (4.13). Moreover, the $W^{1,\infty}$-estimate (3.7) is satisfied by Lemma 2.13. Thus, Proposition 3.8 applies. We find a positive sequence $\nu_n \to 0$ and functions $\varphi_n \in H^1(B_1 \setminus B_{1-\nu_n}, \mathbf{S}_0)$ which satisfy

$$\varphi_n(x) = u_n(x), \qquad \varphi_n(x - \nu_n x) = u_*(x)$$

for $\mathscr{H}^2$-a.e. $x \in \partial B_1$ and

$$E_{\epsilon_n}(\varphi_n,\, B_1 \setminus B_{1-\nu_n}) \leq C\nu_n. \tag{4.14}$$

Now, let $w_* \in H^1(B_1,\, \mathscr{N})$ be a minimizing harmonic extension of $u_{*|\partial B_1}$, i.e.

$$\frac{1}{2}\int_{B_1} |\nabla w_*|^2 \leq \frac{1}{2}\int_{B_1} |\nabla w|^2 \tag{4.15}$$

for any $w \in H^1(B_1,\, \mathscr{N})$ such that $w_{|\partial B_1} = u_{*|\partial B_1}$. Such a function exists by classical results (see e.g. [56, Proposition 3.1]). Define $w_n \colon B_1 \to \mathbf{S}_0$ by

$$w_n(x) := \begin{cases} \varphi_n(x) & \text{if } x \in B_1 \setminus B_{1-\nu_n} \\ w_*\left(\dfrac{x}{1-\nu_n}\right) & \text{if } x \in B_{1-\nu_n}. \end{cases}$$

The function w_n is an admissible comparison function for u_n, i.e. $w_n \in H^1(B_1,\, \mathbf{S}_0)$ and $w_{n|\partial B_1} = u_{n|\partial B_1}$. Hence,

$$E_{\epsilon_n}(u_n,\, B_1) \leq E_{\epsilon_n}(w_n,\, B_1) = \frac{1-\nu_n}{2}\int_{B_1} |\nabla w_*|^2 + E_{\epsilon_n}(w_n,\, B_1 \setminus B_{1-\nu_n}).$$

When we take the limit as $n \to +\infty$, $\nu_n \to 0$ and the energy in the shell $B_1 \setminus B_{1-\nu_n}$ converges to 0, due to (4.14). Keeping (4.10) in mind, we obtain

$$\begin{aligned} \frac{1}{2}\int_{B_1} |\nabla u_*|^2 &\leq \liminf_{n\to+\infty} \frac{1}{2}\int_{B_1} |\nabla u_n|^2 \leq \limsup_{n\to+\infty} \frac{1}{2}\int_{B_1} |\nabla u_n|^2 \\ &\leq \limsup_{n\to+\infty} E_{\epsilon_n}(u_n,\, B_1) \leq \frac{1}{2}\int_{B_1} |\nabla w_*|^2 \leq \frac{1}{2}\int_{B_1} |\nabla u_*|^2, \end{aligned}$$

where the last inequality follows by the minimality of w_*, (4.15). But this implies

$$\lim_{n\to+\infty} \frac{1}{2}\int_{B_1} |\nabla u_n|^2 = \frac{1}{2}\int_{B_1} |\nabla u_*|^2,$$

which yields the strong H^1 convergence $u_n \to u_*$, as well as

$$\lim_{n\to+\infty} \frac{1}{\epsilon_n}\int_{B_1} f(u_n) = 0. \tag{4.16}$$

Moreover, u_* must be a minimizing harmonic map.

Scaling back to Q_{ε_n}, Q_0, we have shown that $Q_{\varepsilon_n} \to Q_0$ strongly in $H^1(B_r,\, \mathbf{S}_0)$ and that Q_0 is minimizing harmonic in B_r, where $r \geq \theta R$. In particular, the proposition holds true. □

Once Proposition 4.3 is established, Proposition 4.4 can be proved arguing as in Majumdar and Zarnescu's paper [47]. As a byproduct of the previous proof (see Equation (4.16)), we obtain the condition

$$\lim_{n\to+\infty} \frac{1}{\varepsilon_n^2}\int_{B_{\theta R}(x_0)} f(Q_{\varepsilon_n}) = 0,$$

which is involved in Majumdar and Zarnescu's arguments (see, in particular, [47, Proposition 4]).

4.3 The singular set

In this subsection, we complete the proof of Theorem 1 by defining the singular set $\mathscr{S}_{\mathrm{line}}$ and studying its properties. Throughout the subsection, we assume that Condition (H) holds. For each $0 < \varepsilon < 1$, define the measure μ_ε by

$$\mu_\varepsilon(B) := \frac{E_\varepsilon(Q_\varepsilon,\, B)}{|\log \varepsilon|} \qquad \text{for } B \in \mathscr{B}(\Omega). \tag{4.17}$$

In view of (H), the measures $\{\mu_\varepsilon\}_{0<\varepsilon<1}$ have uniformly bounded mass. Therefore, we may extract a subsequence $\varepsilon_n \searrow 0$ such that

$$\mu_{\varepsilon_n} \rightharpoonup^\star \mu_0 \qquad \text{weakly}^\star \text{ in } \mathscr{M}_{\mathrm{b}}(\Omega) := C_0(\Omega)'. \tag{4.18}$$

Set $\mathscr{S}_{\mathrm{line}} := \operatorname{supp} \mu_0$. By definition, $\mathscr{S}_{\mathrm{line}}$ is a relatively closed subset of Ω. Let η be given by Proposition 7, corresponding to the choice $\theta = 1/2$.

Lemma 4.5. *Let $R_0 > 0$, $x_0 \in \Omega$ be such that $\overline{B}_R(x_) \subset \Omega$, and let $0 < R \leq R_0$. If*

$$\mu_0\left(\overline{B}_R(x_0)\right) < \eta R \tag{4.19}$$

then

$$\mu_0\left(B_{R/2}(x_0)\right) = 0,$$

that is $B_{R/2}(x_0) \subseteq \Omega \setminus \mathscr{S}_{\mathrm{line}}$.

Proof. In force of (4.18) and (4.19), we know that

$$\limsup_{n\to+\infty} \frac{E_{\varepsilon_n}(Q_{\varepsilon_n},\, B_R(x_0))}{R \log\left(\varepsilon_n/R\right)} < \eta.$$

In particular, the assumption (10) is satisfied along the subsequence $\{\varepsilon_n\}$. Then, we can apply Proposition 7 with $\theta = 1/2$ and we obtain

$$E_{\varepsilon_n}(B_{R/2}(x_0)) \leq CR$$

for n large enough. Due to (4.18), we deduce

$$\mu_0\left(B_{R/2}(x_0)\right) \leq \liminf_{n\to+\infty} \mu_{\varepsilon_n}\left(B_{R/2}(x_0)\right) = 0. \qquad \square$$

By the monotonicity formula (Lemma 2.14), for any $x \in \Omega$ the function

$$r \in (0,\, \operatorname{dist}(x,\, \partial\Omega)) \mapsto \frac{\mu_0\left(\overline{B}_r(x)\right)}{2r}$$

is non-decreasing, so the limit

$$\Theta(x) := \lim_{r\to 0^+} \frac{\mu_0\left(\overline{B}_r(x)\right)}{2r} \tag{4.20}$$

exists. The function Θ is usually called (1-dimensional) density of μ_0 (see [57, p. 10]).

Lemma 4.6. *For all $x \in \mathscr{S}_{\mathrm{line}}$, $\Theta(x) \geq \eta/2$.*

Proof. This follows immediately by Lemma 4.5. Indeed, if $x \in \mathscr{S}_{\mathrm{line}}$ then for any $r > 0$ we have $\mu_0(B_r(x)) > 0$, so Lemma 4.5 implies

$$\frac{\mu_0(\overline{B}_{2r}(x))}{4r} \geq \frac{\eta}{2}.$$

Passing to the limit as $r \to 0$, we conclude. $\square$

Although elementary, this fact has remarkable consequences.

Proposition 4.7. *The set $\mathscr{S}_{\mathrm{line}}$ is countably $\mathscr{H}^1$-rectifiable, with $\mathscr{H}^1(\mathscr{S}_{\mathrm{line}}) < +\infty$. Moreover, the measure μ_0 can be written as $\mu_0 = \Theta\mathscr{H}^1 \llcorner \mathscr{S}_{\mathrm{line}}$, that is*

$$\mu_0(B) = \int_{B\cap\mathscr{S}_{\mathrm{line}}} \Theta(x)\,\mathrm{d}\mathscr{H}^1(x) \qquad \textit{for all } B \in \mathscr{B}(\Omega).$$

Proof. Lemma 4.6, together with [57, Theorem 3.2.(i), Chapter 1] and (H), implies

$$\mathscr{H}^1(\mathscr{S}_{\mathrm{line}}) \leq 2\eta^{-1}\mu_0(\Omega) \leq 2\eta^{-1}M < +\infty.$$

Moreover, since the 1-dimensional density of μ_0 exists and is positive μ_0-a.e., the support of μ_0 is a $\mathscr{H}^1$-rectifiable set and μ_0 is absolutely continuous with respect to $\mathscr{H}^1 \llcorner \mathscr{S}_{\mathrm{line}}$. This fact was proved by Moore [50] and is a special case of Preiss' theorem [52, Theorem 5.3], which holds true for measures in $\mathbb{R}^n$ having positive k-dimensional density, for any $k \leq n$. Thus, there exists a positive, $\mathscr{H}^1$-integrable function $g\colon \Omega \to \mathbb{R}$ such that

$$\mu_0(B) = \int_{B\cap\mathscr{S}_{\mathrm{line}}} g(x)\,\mathrm{d}\mathscr{H}^1(x) \qquad \text{for all } B \in \mathscr{B}(\Omega).$$

By Besicovitch differentiation theorem, there holds

$$\lim_{r\to 0^+} \frac{\mu_0(\overline{B}_r(x))}{\mathscr{H}^1(B_r(x)\cap\mathscr{S}_{\mathrm{line}})} = g(x) \qquad \text{for } \mathscr{H}^1\text{-a.e. } x \in \mathscr{S}_{\mathrm{line}}.$$

On the other hand, because $\mathscr{S}_{\mathrm{line}}$ is rectifiable and $\mathscr{H}^1(\mathscr{S}_{\mathrm{line}}) < +\infty$, Besicovitvh-Marstrand-Mattila theorem (see e.g. [4, Theorem 2.63]) implies that

$$\lim_{r\to 0^+} \frac{\mathscr{H}^1(B_r(x)\cap\mathscr{S}_{\mathrm{line}})}{2r} = 1 \qquad \text{for } \mathscr{H}^1\text{-a.e. } x \in \mathscr{S}_{\mathrm{line}}.$$

By combining these facts and (4.20), we obtain $\Theta = g$ $\mathscr{H}^1$-a.e. on $\mathscr{S}_{\mathrm{line}}$, so the proposition follows. □

The monotonicity of the energy, established in Lemma 2.14, provides a lower bound for the Hausdorff dimension of the singular set $\mathscr{S}_{\mathrm{line}}$.

Lemma 4.8. *For any open set $K \subset\subset \Omega$, either $\mathscr{S}_{\mathrm{line}} \cap K = \emptyset$ or the Hausdorff dimension of $\mathscr{S}_{\mathrm{line}} \cap K$ is 1.*

Proof. If $\mu_0(K) = 0$ then $\mathscr{S}_{\mathrm{line}} \cap K = \emptyset$ and the lemma is proved. Now, we assume that $\mu_0(K) > 0$. By Proposition 4.7 we know that $\mathscr{H}^1(\mathscr{S}_{\mathrm{line}} \cap K) < +\infty$, so the dimension of $\mathscr{H}^1(\mathscr{S}_{\mathrm{line}} \cap K)$ is at most 1. To check that it is exactly equal to 1, it suffices to show that $\mathscr{H}^1(\mathscr{S}_{\mathrm{line}} \cap K) > 0$. Fix $0 < r_0 < \mathrm{dist}(K, \partial\Omega)$. By the monotonicity formula (Lemma 2.14) and the assumption (H), we have

$$\frac{E_\varepsilon(Q_\varepsilon,\, B_r(x))}{2r} \leq \frac{E_\varepsilon(Q_\varepsilon,\, B_{r_0}(x))}{2r_0} \leq \frac{M}{2r_0}\left(|\log\varepsilon| + 1\right)$$

for any $0 < r < r_0$ and $x \in K$. Passing to the limit as $\varepsilon \to 0$, owning to (4.18) we have

$$\frac{\mu_0(\overline{B}_{r/2}(x))}{r} \leq \frac{\mu_0(B_r(x))}{r} \leq \frac{M}{r_0}$$

and, in the limit as $r \to 0^+$, we obtain $\Theta(x) \leq M r_0^{-1}$ for any $x \in K$. Then, [57, Theorem 3.2.(2)] implies

$$\mathscr{H}^1(\mathscr{S}_{\mathrm{line}} \cap K) \geq \frac{r_0}{2M}\mu_0(\mathscr{S}_{\mathrm{line}} \cap K) > 0.$$ □

To complete the proof of Theorem 1, we check that Q_{ε_n} locally converge to a harmonic map, away from $\mathscr{S}_{\mathrm{line}}$.

Proposition 4.9. *There exists a map $Q_0 \in H^1_{\mathrm{loc}}(\Omega \setminus \mathscr{S}_{\mathrm{line}}, \mathscr{N})$ such that, up to a relabeled subsequence,*

$$Q_{\varepsilon_n} \to Q_0 \qquad \text{stongly in } H^1_{\mathrm{loc}}(\Omega \setminus \mathscr{S}_{\mathrm{line}}, \mathbf{S}_0).$$

The map Q_0 is minimizing harmonic on every ball $B \subset\subset \Omega \setminus \mathscr{S}_{\mathrm{line}}$. Moreover, there exists a locally finite set $\mathscr{S}_{\mathrm{pts}} \subseteq \Omega \setminus \mathscr{S}_{\mathrm{line}}$ such that Q_0 is of class C^∞ on $\Omega \setminus (\mathscr{S}_{\mathrm{line}} \cup \mathscr{S}_{\mathrm{pts}})$, and

$$Q_{\varepsilon_n} \to Q_0 \qquad \text{locally uniformly in } \Omega \setminus (\mathscr{S}_{\mathrm{line}} \cup \mathscr{S}_{\mathrm{pts}}).$$

Proof. Let $\{K^p\}_{p\in\mathbb{N}}$ be an increasing sequence of subsets $K^p \subset\subset \Omega \setminus \mathscr{S}_{\mathrm{line}}$, such that $K^p \nearrow \Omega \setminus \mathscr{S}_{\mathrm{line}}$. For each $p \in \mathbb{N}$, the compactness of K^p implies that there exists a finite covering of K^p with balls $\{B(x_i^p, r_i^p)\}_{1\leq i\leq I_p}$ such that

$$\overline{B}(x_i^p, 4r_i^p) \subseteq \Omega \setminus \mathscr{S}_{\mathrm{line}} \qquad \text{i.e.} \qquad \mu_0\left(\overline{B}(x_i^p, 4r_i^p)\right) = 0. \tag{4.21}$$

Due to (4.18), this implies

$$\limsup_{n\to+\infty} \frac{E_{\varepsilon_n}(Q_{\varepsilon_n}, B(x_i^p, 4r_i^p))}{r_i^p \log(\varepsilon_n/r_i^p)} = 0$$

for each i, p. In particular, Condition (10) is satisfied for n large enough. Applying Proposition 7 with $\theta = 1/2$, we infer

$$E_{\varepsilon_n}(Q_{\varepsilon_n}, B(x_i^p, 2r_i^p)) \leq C = C(i,p).$$

By Proposition 4.3 we deduce that, up to a relabeled subsequence, Q_{ε_n} converges strongly in H^1 to a map $Q_0 \in H^1(B(x_i^p, r_i^p), \mathscr{N})$. This is true for every $i \in \{1, \ldots, I_p\}$ so, after a further extraction of subsequences, we obtain

$$Q_{\varepsilon_n} \to Q_0 \quad \text{strongly in } H^1(K^p, \mathbf{S}_0), \quad \text{for all } p \in \mathbb{N}. \tag{4.22}$$

A priori, the subsequence $\{Q_{\varepsilon_n}\}$ depends on p, but one can use a diagonal argument to ensure that (4.22) is satisfied by *the same* subsequence, for all $p \in \mathbb{N}$.

For each ball $B \subset\subset \Omega\setminus\mathscr{S}_{\mathrm{line}}$, the map Q_0 is minimizing harmonic on B. Indeed, fix a larger concentric ball B', with $B \subset\subset B' \subset\subset \Omega\setminus\mathscr{S}_{\mathrm{line}}$. Denote by r, r' the radii of B, B' respectively. Because of $\mu_0(B') = 0$ and (4.18), one has

$$\limsup_{n\to+\infty} \frac{E_{\varepsilon_n}(Q_{\varepsilon_n}, B'}{r' \log(\varepsilon_n/r')} = 0.$$

As before, one applies Proposition 7, then Proposition 4.3, and obtains that Q_0 is minimizing harmonic on a ball of radius $\theta^2 r'$, for an arbitrarily fixed $0 < \theta < 1$. Taking θ so large that $\theta^2 r' > r$, it follows that Q_0 is minimizing harmonic on B.

Thanks to Schoen and Uhlenbeck's partial regularity result [55, Theorem II], we know that on each ball $B \subset\subset \Omega \setminus \mathscr{S}_{\mathrm{line}}$ there exists a finite set $X_B \subseteq B$ such that $Q_0 \in C^\infty(B \setminus X_B, \mathbf{S}_0)$. Therefore, $Q_0 \in C^\infty(\Omega \setminus \mathscr{S}_{\mathrm{line}} \cup \mathscr{S}_{\mathrm{pts}})$, where $\mathscr{S}_{\mathrm{pts}} := \cup_B X_B$ is locally finite in $\Omega \cup \mathscr{S}_{\mathrm{line}}$. The locally uniform convergence $Q_{\varepsilon_n} \to Q_0$ on $\Omega \setminus (\mathscr{S}_{\mathrm{line}} \cup \mathscr{S}_{\mathrm{pts}})$ follows by Proposition 4.4, combined with a covering argument. □

We conclude our discussion about the properties of the singular set by proving that μ_0 is a stationary varifold. This will prove Proposition 2. These objects, introduced by Almgren [3], can be thought as weak counterparts of manifolds with vanishing mean curvature. For more details, the reader is referred to the paper by Allard [2] or the book by Simon [57]. Proposition 4.7 implies that μ_0 is a rectifiable varifold; using the notation of [57, Chapter 4], we have $\mu_0 = \mu_{\mathbf{V}_0}$, where $\mathbf{V}_0 := \mathbf{V}(\mathscr{S}_{\mathrm{line}}, \Theta)$.

Before stating the following proposition, let us recall a basic fact. The rectifiability of μ_0, together with [57, Remarks 1.9 and 11.5, Theorem 11.6], implies that for μ_0-a.e. x there exists a unique 1-dimensional subspace $L_x \subseteq \mathbb{R}^n$ such that

$$\lim_{\lambda\to 0}\int_{\mathbb{R}^d} \lambda^{-1}\varphi\left(\frac{z-x}{\lambda}\right) \mathrm{d}\mu_0(z) = \Theta(x)\int_{L_x} \varphi(y)\,\mathrm{d}\mathscr{H}^1(y) \qquad \text{for all } \varphi\in C_c(\mathbb{R}^3). \tag{4.23}$$

Such line is called the *approximate tangent line* of μ_0 at x, and noted $\mathrm{Tan}(\mu_0, x)$.

Proposition 4.10. *The varifold* $\mathbf{V}_0$ *is stationary, i.e. for any vector field* $X \in C^1_c(\Omega, \mathbb{R}^3)$ *there holds*

$$\int_\Omega A_{ij}(x)\frac{\partial X^i}{\partial x_j}(x)\,\mathrm{d}\mu_0(x) = 0,$$

where the matrix $A(x)\in \mathrm{M}_3(\mathbb{R})$ *represents the orthogonal projection on* $\mathrm{Tan}(\mu_0, x)$*, for all* $x\in\Omega$.

Proof. The proposition follows by adapting Ambrosio and Soner's analysis in [5]. For the convenience of the reader, we give here the proof. Define the matrix-valued map $A^\varepsilon = (A^\varepsilon_{ij})_{i,j}\colon \Omega\to \mathrm{M}_3(\mathbb{R})$ by

$$A^\varepsilon_{ij} := \frac{1}{|\log\varepsilon|}\left(e_\varepsilon(Q_\varepsilon)\delta_{ij} - \frac{\partial Q_\varepsilon}{\partial x_i}\cdot\frac{\partial Q_\varepsilon}{\partial x_j}\right) \qquad \text{for } i, j\in\{1,\,2,\,3\}.$$

Then A^ε is a symmetric matrix, such that

$$\mathrm{tr}\, A^\varepsilon = \frac{1}{|\log\varepsilon|}\left(3e_\varepsilon(Q_\varepsilon) - |\nabla Q_\varepsilon|^2\right) \geq \mu_\varepsilon \tag{4.24}$$

and

$$|A^\varepsilon| \leq C\mu_\varepsilon. \tag{4.25}$$

For any vector $v\in\mathbb{S}^2$, there holds

$$A^\varepsilon_{ij}v_iv_j = \frac{1}{|\log\varepsilon|}\left(e_\varepsilon(Q_\varepsilon) - \left|v_i\frac{\partial Q_\varepsilon}{\partial x_i}\right|^2\right) \leq \mu_\varepsilon, \tag{4.26}$$

so the eigenvalues of A^ε are less or equal than μ_ε. Moreover, by integrating by parts the stress-energy identity (Lemma 2.15) we obtain

$$\int_\Omega A^\varepsilon_{ij}(x)\frac{\partial X^i}{\partial x_j}(x)\,\mathrm{d}x = 0 \qquad \text{for any } X\in C^1_c(\Omega,\,\mathbb{R}^3). \tag{4.27}$$

In view of (4.25), and extracting a subsequence if necessary, we have that $A^\varepsilon \rightharpoonup^\star A^0$ in the weak-$\star$ topology of $\mathscr{M}(\Omega,\,\mathrm{M}_3(\mathbb{R})) = C_c(\Omega,\,\mathrm{M}_3(\mathbb{R}))'$. The limit measure A^0 satisfies $|A^0|\leq C\mu_0$, in particular is absolutely continuous with respect to μ_0. Therefore, there exists a matrix-valued function $A\in L^1(\Omega,\,\mu_0;\,\mathrm{M}_3(\mathbb{R}))$ such that

$$\mathrm{d}A^0(x) = A(x)\mathrm{d}\mu_0(x) \qquad \text{as measures in } \mathscr{M}(\Omega,\,\mathrm{M}_3(\mathbb{R})).$$

Passing to the limit in (4.24), (4.26) and (4.27), for μ_0-a.e. x we obtain that $A(x)$ is a symmetric matrix, with $\mathrm{tr}\,A(x)\geq 1$ and eigenvalues less or equal than 1, such that

$$\int_\Omega A_{ij}(x)\frac{\partial X^i}{\partial x_j}(x)\mathrm{d}\mu_0(x) = 0 \qquad \text{for any } X\in C^1_c(\Omega,\,\mathbb{R}^3). \tag{4.28}$$

Now, fix a Lebesgue point x for A (with respect to μ_0) and $0 < \lambda < \mathrm{dist}(x, \partial\Omega)$. Condition (4.28) implies

$$\lambda^{-1} \int_{\mathbb{R}^3} A(z) \cdot \nabla X \left(\frac{z-x}{\lambda}\right) \mathrm{d}\mu_0(z) = 0 \qquad \text{for any } X \in C^1_c(B_1, \mathbb{R}^3). \tag{4.29}$$

Then,

$$\begin{aligned} &\left|\lambda^{-1} \int_{\mathbb{R}^3} (A(z) - A(x)) \cdot \nabla X \left(\frac{z-x}{\lambda}\right) \mathrm{d}\mu_0(z)\right| \\ &\qquad \leq \underbrace{\frac{\mu_0(\overline{B}_\lambda(x))}{\lambda}}_{\to \Theta(x)/2} \|\nabla X\|_{L^\infty(B_1)} \fint_{B_\lambda(x)} |A(z) - A(x)| \, d\mu_0(z) \to 0 \end{aligned}$$

as $\lambda \to 0$. Combined with (4.23) and (4.29), this provides

$$\Theta(x) A(x) \cdot \int_{\mathrm{Tan}(\mu_0, x)} \nabla X \, \mathrm{d}\mathscr{H}^1 = \lim_{\lambda \to 0} \lambda^{-1} \int_{\mathbb{R}^3} A(x) \cdot \nabla X \left(\frac{z-x}{\lambda}\right) \mathrm{d}\mu_0(x) = 0$$

for any $X \in C^1_c(B_1, \mathbb{R}^3)$. Since $\Theta(x) > 0$ by Lemma 4.6, applying [5, Lemma 3.9] (with $\beta = s = 1$ and $\nu = \frac{1}{2}\mathscr{H}^1 \llcorner \mathrm{Tan}(\mu_0, x)$) we deduce that at least two eigenvalues of $A(x)$ vanish, for μ_0-a.e. x. On the other hand, we know already that $\mathrm{tr}\, A(x) = 1$ with eigenvalues ≤ 1. Therefore, the eigenvalues of $A(x)$ are $(1, 0, 0)$ and $A(x)$ represents the orthogonal projection on a line.

Let $\mathbf{G}_{1,3} \subseteq \mathrm{M}_3(\mathbb{R})$ be the set of matrices representing orthogonal projections on 1-subspaces of $\mathbb{R}^3$. The push-forward measure $\mathbf{V} := (\mathrm{Id}, A)_{\#}\mu_0$, i.e. the measure $\mathbf{V} \in \mathscr{M}(\Omega \times \mathbf{G}_{1,3})$ given by

$$\int_{\Omega \times \mathbf{G}_{1,3}} \varphi(x, M) \, \mathrm{d}\mathbf{V}(x, M) := \int_\Omega \varphi(x, A(x)) \, \mathrm{d}\mu_0(x) \qquad \text{for } \varphi \in C_c(\Omega \times \mathbf{G}_{1,3}), \tag{4.30}$$

is a varifold in the sense of Almgren, and condition (4.28) means precisely that $\mathbf{V}$ is stationary. A classical result by Allard (see [2] or [5, Theorem 3.3]) asserts that every varifold with locally bounded first variation and positive density is rectifiable. In our case, $\mathbf{V}$ has vanishing first variation, and the density is bounded from below by Lemma 4.6. Therefore, by Allard's theorem $\mathbf{V}$ is rectifiable. In particular $A(x)$ is the orthogonal projection on $\mathrm{Tan}(\mathscr{S}_{\mathrm{line}}, x)$, for μ_0-a.e. x. □

4.4 The analysis near the boundary

Proposition 7, which is the key step in the proof of our main theorem, has been proven on balls included in the interior of the domain. In this subsection, we aim at proving a similar result in case the ball intersect the boundary of Ω. For this purpose, we need an additional assumption on the behaviour of the boundary datum. Let Γ be a relatively open subset of $\partial\Omega$. We assume that

(H'_Γ) $g_\varepsilon \in (H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}})(\Gamma, \mathbf{S}_0)$ for any ε and, for any $K \subset\subset \Gamma$, there exists a constant C_K such that

$$E_\varepsilon(g_\varepsilon, K) \leq C_K \qquad \text{and} \qquad \|g_\varepsilon\|_{L^\infty(K)} \leq C_K$$

for any $0 < \varepsilon < 1$.

For instance, the families of boundary data given by (8) and (9) satisfies Condition (H'_Γ) on $\Gamma := \partial\Omega \setminus \Sigma$.

Proposition 4.11. *Assume that the conditions* (H) *and* (H'_Γ) *hold. For any* $0 < \theta < 1$ *there exist positive numbers* η, ϵ_0 *and* C *such that, for any* $x_0 \in \overline{\Omega}$, $R > 0$ *satisfying* $\overline{B}_R(x_0) \cap \partial\Omega \subseteq \Gamma$ *and any* $0 < \varepsilon \leq \epsilon_0 \theta R$, *if*

$$E_\varepsilon(Q_\varepsilon, B_R(x_0) \cap \Omega) \leq \eta R \log \frac{R}{\varepsilon} \tag{4.31}$$

then

$$E_\varepsilon(Q_\varepsilon,\, B_{\theta R}(x_0)\cap\Omega)\leq CR.$$

By a standard covering argument, similar to that employed in the proof of Proposition 4.9, we see that Proposition 4.11 implies weak compactness of minimizers up to the boundary. More precisely, we have

Corollary 4.12. *Let Γ be a relatively open subset of $\partial\Omega$. Assume that the conditions* (H) *and* (H'_Γ) *are satisfied. Then, there exist a subsequence $\varepsilon_n \searrow 0$, a closed set $\mathscr{S}_{\mathrm{line}}\subseteq\overline{\Omega}$ and a map $Q_0\in H^1_{\mathrm{loc}}((\Omega\cup\Gamma)\setminus \mathscr{S}_{\mathrm{line}},\,\mathscr{N})$ which satisfy (i)–(v) in Theorem 1 and*

$$Q_{\varepsilon_n}\rightharpoonup Q_0 \qquad \textit{weakly in } H^1_{\mathrm{loc}}((\Omega\cup\Gamma)\setminus\mathscr{S}_{\mathrm{line}},\,\mathbf{S}_0).$$

The set $\mathscr{S}_{\mathrm{line}}$ is again defined as the support of the measure μ_0, where μ_0 is a weak* limit of $\{\mu_\varepsilon\}_{0<\varepsilon<1}$ in $C(\overline{\Omega})'$ and the μ_ε's are given by (4.17). The proofs in Subsection 4.3 remain unchanged. We cannot expect strong H^1 convergence of minimizers up to the boundary, unless some additional assumption on the boundary datum is made. Moreover, we do *not* expect that $\mathscr{S}_{\mathrm{line}}\cap\Gamma=\emptyset$ in general (see Remark 5.1).

Proof of Proposition 4.11. For the sake of simplicity, we assume that $x_0=0$ and set $F_\varepsilon(r):=E_\varepsilon(Q_\varepsilon,\, B_r\cap\Omega)$ for $0<r<R$. The coarea formula implies

$$F_\varepsilon(r)=\int_0^r E_\varepsilon(Q_\varepsilon,\,\partial B_s\cap\Omega)\,\mathrm{d}s \qquad \text{for } 0<r<R,$$

so $F'_\varepsilon(r)=E_\varepsilon(Q_\varepsilon,\,\partial B_r\cap\Omega)$ for a.e. $0<r<R$. Define the set

$$\tilde{D}^\varepsilon := \left\{ r\in(\theta R,\, R)\colon F'_\varepsilon(r)\leq \frac{2\eta}{1-\theta}\log\frac{R}{\varepsilon}\right\}.$$

The assumption (4.31) and an average argument give

$$\mathscr{H}^1(\tilde{D}^\varepsilon)\geq\frac{(1-\theta)R}{2}. \tag{4.32}$$

On the other hand, for any radius $r\in\tilde{D}^\varepsilon$ we have

$$E_\varepsilon(Q_\varepsilon,\,\partial(B_r\cap\Omega))=F'_\varepsilon(r)+E_\varepsilon(Q_\varepsilon,\,B_r\cap\partial\Omega)\overset{(\mathrm{H}'_\Gamma)}{\leq}\frac{2\eta}{1-\theta}\left(\log\frac{r}{\varepsilon}-\log\theta\right)+C,$$

where C is a constant depending on x_0 and R. Therefore, by choosing η small enough we obtain

$$E_\varepsilon(Q_\varepsilon,\,\partial(B_r\cap\Omega))\leq\eta_0\log\frac{r}{\varepsilon} \qquad \text{for } 0<\varepsilon\leq\epsilon_0\theta R,$$

where η_0 and ϵ_0 are given by Proposition 3.6. With the help of this estimate, and since $B_r\cap\Omega$ is bilipschitz equivalent to a ball, we can repeat the proof of Lemma 4.1. We deduce that

$$F_\varepsilon(r)\leq CR\left(E_\varepsilon(Q_\varepsilon,\,\partial(B_r\cap\Omega))^{1/2}+1\right) \qquad \text{for any } r\in\tilde{D}^\varepsilon \text{ and } 0<\varepsilon\leq\epsilon_0\theta R.$$

Then, using the elementary inequality $(a+b)^{1/2}\leq a^{1/2}+b^{1/2}$ and (H'_Γ) again, we infer

$$F_\varepsilon(r)\leq CR\left\{\left(F'_\varepsilon(r)+E_\varepsilon(Q_\varepsilon,\,B_r\cap\partial\Omega)\right)^{1/2}+1\right\}\leq CR\left(F'_\varepsilon(r)^{1/2}+1\right) \tag{4.33}$$

for any $r\in\tilde{D}^\varepsilon$ and $0<\varepsilon\leq\epsilon_0\theta R$. Thanks to (4.32) and (4.33), we can apply Lemma 4.2 to $y:=F_\varepsilon$. This yields the conclusion of the proof. □

5 Sufficient conditions for (H). The role of the boundary data

5.1 Proof of Propositions 4 and 5

In this section, we analyze the role of the domain and the boundary data in connection with (H), and prove sufficient conditions for (H) to hold true. We prove first Proposition 4, namely, we show that the assumption (H_2) on the topology of Ω combined with the logarithmic upper bound (H_3) on the energy of the boundary data imply

$$\|Q_\varepsilon\|_{L^\infty(\Omega)} \leq M \tag{5.1}$$

and

$$E_\varepsilon(Q_\varepsilon) \leq M\left(|\log\varepsilon| + 1\right), \tag{5.2}$$

for some positive constant $M = M(\Omega,\, M_0)$. At the end of the subsection, we also prove Proposition 5.

Lemma 5.1. *Minimizers Q_ε of* (LG_ε) *satisfy*

$$\|Q_\varepsilon\|_{L^\infty(\Omega)} \leq \max\left\{\sqrt{\frac{2}{3}}s_*,\, \|g_\varepsilon\|_{L^\infty(\partial\Omega)}\right\}.$$

Proof. Set

$$M := \max\left\{\sqrt{\frac{2}{3}}s_*,\, \|Q_\varepsilon\|_{L^\infty(\partial\Omega)}\right\},$$

and define $\pi\colon \mathbf{S}_0 \to \mathbf{S}_0$ by $\pi(Q) := MQ/|Q|$ if $|Q| \geq M$, $\pi(Q) := Q$ otherwise. We have

$$\mathrm{D}f(Q)\cdot Q = -a\,|Q|^2 - b\operatorname{tr}Q^3 + c\,|Q|^4 > 0 \qquad \text{when } |Q| > \sqrt{\frac{2}{3}}s_*$$

(this follows from the inequality $\sqrt{6}|\operatorname{tr}Q^3| \leq |Q|^3$; see [46, Lemma 1]). We deduce that $f(\pi(Q)) \geq f(Q)$ for any Q. Moreover, π is the projection on a convex set, so it is 1-Lipschitz continuous. Thus, the map $\pi\circ Q_\varepsilon$ belongs to $H^1_{g_\varepsilon}(\Omega,\, \mathbf{S}_0)$, satisfies $|\nabla(\pi\circ Q_\varepsilon)| \leq |\nabla Q_\varepsilon|$ a.e. and $E_\varepsilon(\pi\circ Q_\varepsilon) \leq E_\varepsilon(Q_\varepsilon)$, with strict inequality if $|Q_\varepsilon| > M$ on a set of positive measure. By minimality of Q_ε, we conclude that $|Q_\varepsilon| \leq M$ a.e. □

Now, we prove the energy bound (5.2) by constructing an admissible comparison function whose energy is controlled by the right-hand side of (5.2). If Ω is a ball, it suffices to extend homogeneously the boundary data, thanks to (H_3). Since Ω is bilipschitz equivalent to a handlebody by (H_2), we reduce to the case of a ball by cutting each handle of Ω along a meridian disk. This technique was used already in [34, Lemma 1.1]. The following lemma allow us to extend the boundary datum to the interior of the cut disks. Unlike the results of Section 3, here we do not constrain the extension to take values in $\mathscr{N}$. Instead, we prescribe a logarithmic upper bound for its energy.

Lemma 5.2. *There exists a constant $C > 0$ such that, for any $0 < \varepsilon < 1$ and any function $g \in H^1(\partial B^2_1,\, \mathscr{N})$, there exists $v \in H^1(B^2_1,\, \mathbf{S}_0)$ such that $v_{\varepsilon|\partial B^2_1} = g$ and*

$$E_\varepsilon(v,\, B^2_1) \leq C\left(\int_{\partial B^2_1}|\nabla g|^2\,\mathrm{d}\mathscr{H}^1 + |\log\varepsilon| + 1\right).$$

Proof. In view of the Sobolev embedding $H^1(\partial B^2_1, \mathbf{S}_0) \hookrightarrow C^0(\partial B^2_1, \mathbf{S}_0)$, it makes sense to consider the homotopy class of g. If g is homotopically trivial, it may be extended to a function in $H^1(B^2_1, \mathbf{S}_0)$, still denoted g by simplicity. Then, Lemma 3.2 provides a function v with the desired properties.

Assume now that g is homotopically non-trivial, and fix arbitrarily another non-trivial loop $h \in H^1(\partial B^2_{1/2}, \mathscr{N})$. For instance, one can choose $h(x) = P(2x)$ for $x \in \partial B^2_{1/2}$, where P is given by Lemma 2.11. It is easy to check that the function

$$w_\varepsilon(x) := \eta_\varepsilon(|x|) h\left(\frac{x}{|x|}\right) \qquad \text{for } x \in B^2_{1/2},$$

where

$$\eta_\varepsilon(r) := \begin{cases} 1 & \text{if } r \geq \varepsilon \\ \varepsilon^{-1} r & \text{if } 0 \leq r < \varepsilon, \end{cases} \tag{5.3}$$

belongs to $H^1(B^2_{1/2}, \mathbf{S}_0)$ and

$$E_\varepsilon(w_\varepsilon, B^2_{1/2}) \leq C\left|\log\varepsilon\right| \int_{\partial B^2_{1/2}} \left|\nabla_\top h\right|^2 \,\mathrm{d}\mathscr{H}^1 + C \leq C\left(\left|\log\varepsilon\right| + 1\right). \tag{5.4}$$

Indeed,

$$\left|\nabla w_\varepsilon\right|^2 = \left|\frac{\mathrm{d}w_\varepsilon}{\mathrm{d}r}\right|^2 + \frac{1}{r^2}\left|\nabla_\top w_\varepsilon\right|^2 \begin{cases} \leq C\varepsilon^{-1} & \text{where } r \leq \varepsilon \\ = r^{-2}\left|\nabla_\top h\right|^2 & \text{where } r \geq \varepsilon, \end{cases}$$

and $w_\varepsilon(x) \in \mathscr{N}$ if $|x| \geq \varepsilon$. Therefore, we have

$$\begin{aligned} E_\varepsilon(w_\varepsilon, B^2_{1/2}) &\leq \int_\varepsilon^{1/2} \frac{\mathrm{d}r}{r} \int_{\mathbb{S}^1} \left|\nabla_\top h\right|^2 \,\mathrm{d}\mathscr{H}^1 + E_\varepsilon(w_\varepsilon, B^2_\varepsilon) \\ &\leq \left(\left|\log\varepsilon\right| - \log 2\right) \int_{\mathbb{S}^1} \left|\nabla_\top h\right|^2 \,\mathrm{d}\mathscr{H}^1 + C, \end{aligned}$$

whence (5.4) follows.

To complete the proof of the lemma, we only need to interpolate between g and h by a function defined on the annulus $D := B^2_1 \setminus B^2_{1/2}$. Up to a bilipschitz equivalence, D can be thought as the unit square $(0, 1)^2$ with an equivalence relation identifying two opposite sides of the boundary, as shown in Figure 3. We assign the boundary datum g on the bottom side, and h on the top side. Since $\mathscr{N}$ is path-connected, we find a smooth arc $c\colon [0, 1] \to \mathscr{N}$ connecting $g(0, 0)$ to $h(0, 1)$. By assigning c as a boundary datum on the lateral sides of the square, we have defined an H^1-map $\partial[0, 1]^2 \to \mathscr{N}$, homotopic to $g{*}c{*}h{*}\tilde{c}$. (Here, the symbol $*$ stands for composition of paths, and $\tilde{c}$ is the reverse path of c.) Since the square is bilipschitz equivalent to a disk, it is possible to apply Lemma 3.2 and find $\tilde{v} \in H^1([0, 1]^2, \mathscr{N})$ such that

$$\int_{[0,1]^2} \left|\nabla\tilde{v}\right|^2 \,\mathrm{d}\mathscr{H}^2 \leq C\left(\|\nabla g\|^2_{L^2(\partial B^2_1)} + \|\nabla h\|^2_{L^2(\partial B^2_{1/2})} + \|c'\|^2_{L^2(0,1)}\right). \tag{5.5}$$

Passing to the quotient $[0, 1]^2 \to D$, we obtain a map $v \in H^1(D, \mathbf{S}_0)$. We extend v by setting $v := w_\varepsilon$ on $B^2_{1/2}$. The lemma now follows from (5.4) and (5.5), because the H^1-norms of both h and c are controlled by a constant depending only on $\mathscr{N}$. $\square$

In the following lemma, we construct cut disks with suitable properties.

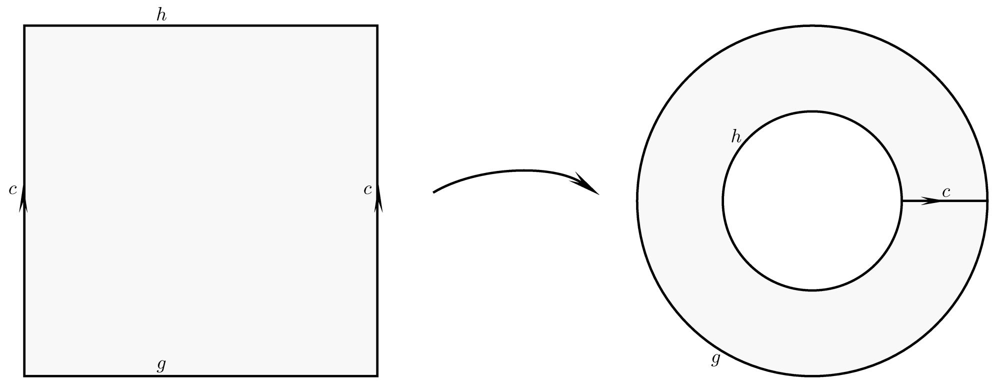


Figure 3: A square can be mapped into an annulus, by identifying a pair of opposite sides.

Lemma 5.3. *Assume that* (H_2) *and* (H_3) *hold. There exists a finite number of properly embedded disks*[3] $D_1, D_2, \ldots, D_k \subseteq \Omega$ *such that* $\Omega \setminus \cup_{i=1}^k D_i$ *is diffeomorphic to a ball,*

$$E_\varepsilon(g_\varepsilon, \partial D_i) \leq C\left(|\log \varepsilon| + 1\right) \tag{5.6}$$

and

$$\operatorname{dist}(g_\varepsilon(x), \mathscr{N}) \to 0 \qquad \textit{uniformly in } x \in \bigcup_{i=1}^k \partial D_i. \tag{5.7}$$

Proof. For each handle i of Ω, there is an open set U_i such that $\partial\Omega \cap U_i$ is foliated by

$$\partial\Omega \cap U_i = \coprod_{-a_0 < a < a_0} \partial D_i^a,$$

where the generic D_i^a is a properly embedded disk, which cross transversely a generator of $\pi_1(\Omega)$ at some point. Then, Fatou's lemma implies that

$$\int_{-a_0}^{a_0} \liminf_{\varepsilon \to 0} E_\varepsilon(g_\varepsilon, \partial D_i^a)\,\mathrm{d}a \leq \liminf_{\varepsilon \to 0} \int_{-a_0}^{a_0} E_\varepsilon(g_\varepsilon, \partial D_i^a)\,\mathrm{d}a \overset{(\mathrm{H}_3)}{\leq} C\left(|\log \varepsilon| + 1\right),$$

so, by an average argument, we can choose the parameter a in such a way that $D_i := D_i^a$ satisfies (5.6). Then, (5.7) is obtained by the same arguments as Lemma 3.10. (As in the lemma, we apply Sobolev-Morrey's embedding inequality not on ∂D_i directly, but on 1-cells $K \subseteq \partial D_i$ of size comparable to $\varepsilon^\alpha |\log \varepsilon|$.) Furthermore, by construction $\Omega \setminus \cup_{i=1}^k D_i$ is a ball, since we have removed a meridian disk for each handle of Ω. $\square$

Proof of Proposition 4. The L^∞-bound (5.1) holds by virtue of Lemma 5.1, so we only need to prove (5.2). Assume for a moment that $\Omega = B_1$. In this case, define the function

$$P_\varepsilon(x) := \eta_\varepsilon(|x|) g_\varepsilon\left(\frac{x}{|x|}\right) \qquad \text{for } x \in B_1, \tag{5.8}$$

[3]By saying that D_i is *properly* embedded, we mean that $\partial D_i = D_i \cap \partial\Omega$ and D_i is transverse to $\partial\Omega$ at each point of ∂D_i.

where η_ε is given by (5.3). Then $P_\varepsilon \in H^1_{g_\varepsilon}(B_1, \mathbf{S}_0)$ and we easily compute

$$\begin{aligned} E_\varepsilon(P_\varepsilon) &= E_\varepsilon(P_\varepsilon, B_1 \setminus B_\varepsilon) + E_\varepsilon(P_\varepsilon, B_\varepsilon) \\ &= \int_\varepsilon^1 \int_{\partial B_1} \left(|\nabla_\top g_\varepsilon|^2 + \varepsilon^{-2} r^2 f(g_\varepsilon)\right) \mathrm{d}\mathscr{H}^2 \, \mathrm{d}r \\ &\qquad + \int_0^\varepsilon \int_{\partial B_1} \varepsilon^{-2} r^2 \left(|g_\varepsilon|^2 + |\nabla_\top g_\varepsilon|^2 + f(P_\varepsilon)\right) \mathrm{d}\mathscr{H}^2 \, \mathrm{d}r. \end{aligned}$$

In view of Assumption (H_3), this yields

$$E_\varepsilon(P_\varepsilon) \leq C \left(E_\varepsilon(g_\varepsilon, \partial B_1) + 1\right) \leq C \left(|\log \varepsilon| + 1\right),$$

so the lemma holds true when $\Omega = B_1$.

Now, arguing as in [34, Lemma 1.1], we prove that the general case can be reduced to the previous one. Let Ω be any domain satisfying (H_2), and let $D_1, \ldots, D_k$ be the disks given by Lemma 5.3. By (5.7), there exists $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon \leq \varepsilon_0$ and any $x \in \cup_i \partial D_i$,

$$\mathrm{dist}(g_\varepsilon(x), \mathscr{N}) \leq \delta_0.$$

For ease of notation, for a fixed $i \in \{1, \ldots, k\}$ we assume, up to a bilipschitz equivalence, that $D_i = B^2_1$. Then, we define $\hat{g}_{\varepsilon,i}\colon B^2_1 \to \mathbf{S}_0$ by

$$\hat{g}_{\varepsilon,i}(x) := \begin{cases} \dfrac{\delta_0 + |x| - 1}{\delta_0} g_\varepsilon\left(\dfrac{x}{|x|}\right) + \dfrac{1 - |x|}{\delta_0} (\varrho \circ g_\varepsilon)\left(\dfrac{x}{|x|}\right) & \text{if } 1 - \delta_0 \leq |x| \leq 1 \\ v_\varepsilon\left(\dfrac{x}{1 - \delta_0}\right) & \text{if } |x| \leq 1 - \delta_0, \end{cases}$$

where $v_\varepsilon \in H^1(B^2_1, \mathbf{S}_0)$ is the extension of $\varrho \circ g_{\varepsilon|\partial B^2_1}$ given by Lemma 5.2. By a straightforward computation, one checks that

$$E_\varepsilon(\hat{g}_{\varepsilon,i}, D_i) \leq C \left(E_\varepsilon(g_\varepsilon, \partial D_i) + |\log \varepsilon| + 1\right). \tag{5.9}$$

Now, consider two copies D^+_i and D^-_i of each disk D_i. Let Ω' be a smooth domain such that

$$\Omega' \simeq (\Omega \setminus \cup_i D_i) \cup_i D^+_i \cup_i D^-_i,$$

and let $\varphi\colon \Omega' \to \Omega$ be the smooth map which identifies each D^+_i with the corresponding D^-_i (see Figure 4). This new domain is simply connected, and in fact is diffeomorphic to a ball. Up to a bilipschitz equivalence, we will assume that Ω' *is* a ball. We define a boundary datum g'_ε for Ω' by setting $g'_\varepsilon := g_\varepsilon$ on $\Omega \setminus \cup_i D_i$, and $g'_\varepsilon := g_{\varepsilon,\,i}$ on $D^+_i \cup D^-_i$. Then, (5.9), (5.6) and (H_3) imply

$$E_\varepsilon(g'_\varepsilon, \partial\Omega) \leq C \left(E_\varepsilon(g_\varepsilon, \partial\Omega) + |\log \varepsilon| + 1\right) \leq C \left(|\log \varepsilon| + 1\right).$$

Then Formula (5.8) gives a map $P'_\varepsilon \in H^1_{g'_\varepsilon}(\Omega', \mathbf{S}_0)$ which satisfies

$$E_\varepsilon(P'_\varepsilon, \Omega') \leq C \left(|\log \varepsilon| + 1\right).$$

Since $P'_{\varepsilon|D^+_i} = P'_{\varepsilon|D^-_i}$ for every i, the map P'_ε factorizes through φ, and defines a new function $P_\varepsilon \in H^1_{g_\varepsilon}(\Omega, \mathbf{S}_0)$ such that

$$E_\varepsilon(P_\varepsilon, \Omega) \leq C \left(|\log \varepsilon| + 1\right).$$

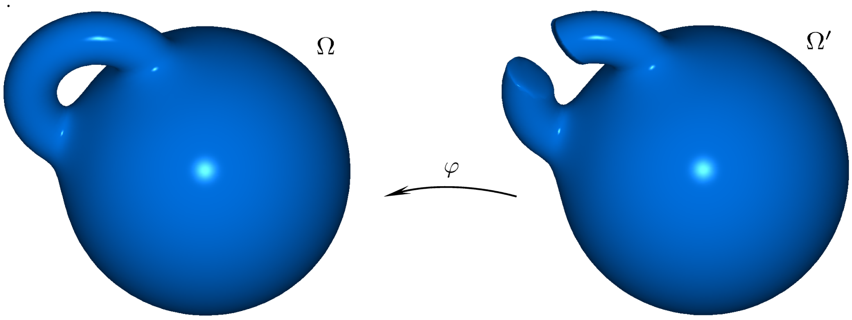


Figure 4: On the left, a ball with one handle. On the right, the corresponding domain Ω': the handle has been cut along a disk. The map $\varphi\colon \Omega' \to \Omega$ identifies the opposite disks in the handle cut.

By comparison, we conclude that (5.2) holds for any $0 < \varepsilon \leq \varepsilon_0$. Now, fix $\varepsilon_0 < \varepsilon < 1$ and consider the ($\mathbf{S}_0$-valued) harmonic extension $\tilde{P}_\varepsilon$ of g_ε. There holds

$$\|\nabla \tilde{P}_\varepsilon\|^2_{L^2(\Omega)} \leq C \, \|\nabla g_\varepsilon\|^2_{L^2(\partial\Omega)} \overset{(\mathrm{H}_3)}{\leq} C \left(|\log \varepsilon| + 1\right), \qquad \|\tilde{P}_\varepsilon\|^2_{L^\infty(\Omega)} \leq C \, \|g_\varepsilon\|^2_{L^\infty(\partial\Omega)} \overset{(\mathrm{H}_3)}{\leq} C$$

and so

$$E_\varepsilon(P_\varepsilon) \leq \frac{1}{2} \|\nabla \tilde{P}_\varepsilon\|^2_{L^2(\Omega)} + C\varepsilon_0^{-2} \leq C \left(1 + \varepsilon_0^2\right) \left(|\log \varepsilon| + 1\right).$$

Also in this case, the lemma follows by comparison. □

We turn now to the proof of Proposition 5. The boundary data we construct are smooth approximations of a map $\partial\Omega \to \mathscr{N}$ with at least one point singularity. Then, the lower bound for the energy follows by the estimates of Subsection 2.2.

Proof of Proposition 5. Up to rotations and translations, we can assume that the x_3-axis $\{x_1 = x_2 = 0\}$ crosses transversely $\partial\Omega$ at one point x_0 at least. Let $\eta_\varepsilon \in C^\infty(\mathbb{R}^+, \, \mathbb{R})$ be a cut-off function satisfying

$$\eta_\varepsilon(0) = \eta_\varepsilon'(0) = 0, \qquad \eta_\varepsilon(r) = s_* \text{ for } r \geq \varepsilon, \qquad 0 \leq \eta_\varepsilon \leq s_*, \qquad |\eta_\varepsilon'| \leq C\varepsilon^{-1}.$$

Set

$$g_\varepsilon(x) := \eta_\varepsilon(|x'|) \left\{ \left(\frac{x'}{|x'|} \right)^{\otimes 2} - \frac{1}{3} \, \mathrm{Id} \right\} \qquad \text{for } x \in \partial\Omega,$$

where $x' := (x_1, \, x_2, \, 0)$. Computing as in Lemma 5.2, one sees that $\{g_\varepsilon\}$ satisfies (H_3). It remains to prove that the energy of minimizers Q_ε satisfies a logarithmic lower bound. Take a ball $B_r(x_0)$. If the radius r is small enough, the set $\Omega \cap B_r(x_0)$ can be mapped diffeomorphically onto the half-ball

$$U := \left\{ x \in \mathbb{R}^3 \colon \, |x| \leq 1, \; x_3 \geq 0 \right\},$$

so we can assume without loss of generality that $\Omega \cap B_r(x_0) = U$. Let $U_s := \{x \in U \colon x_3 = s\}$, for $r/2 \leq s \leq r$. The map $Q_{\varepsilon|\partial U_s} \colon \partial U_s \to \mathscr{N}$ is a homotopically non-trivial loop, which satisfies $E_\varepsilon(Q_\varepsilon, \, \partial U_s) \leq C$. Then, by applying Corollary 2.8 applies we deduce

$$E_\varepsilon(Q_\varepsilon, \, U_s) \geq \kappa_* \log \frac{s}{\varepsilon} - C$$

for a constant C depending on r, Ω. By integrating this bound for $s \in (r/2, \, r)$, the proposition follows. □

Remark 5.1. Let $\{Q_\varepsilon\}$ be a sequence of minimizers and α be a positive number such that

$$E_\varepsilon(Q_\varepsilon) \geq \alpha\left(|\log\varepsilon| - 1\right) \tag{5.10}$$

for any ε, as in Proposition 5. Let μ_ε be the measure defined by (4.17). Then, there exist a subsequence $\varepsilon_n \searrow 0$ and a bounded measure $\mu_0 \in \mathscr{M}_{\mathrm{b}}(\mathbb{R}^3)$ such that

$$\mu_{\varepsilon_n} \rightharpoonup^\star \mu_0 \qquad \text{in } \mathscr{M}_{\mathrm{b}}(\mathbb{R}^3)$$

and $\mu_0(\overline{\Omega}) \geq \alpha > 0$. In particular, the support $\mathscr{S}_{\mathrm{line}}$ of μ_0 is a non-empty, closed subset of $\overline{\Omega}$. However, it might happen that $\mathscr{S}_{\mathrm{line}}$ is contained in the boundary of Ω, even if the boundary datum is regular. For instance, let the domain Ω be a solid torus, parametrized by the map

$$\mathbf{r}\colon (\rho,\,\theta,\,\varphi) \in [0,\,1) \times [0,\,2\pi]^2 \mapsto \begin{cases} x_1 = (2 + \rho\cos\varphi)\cos\theta \\ x_2 = (2 + \rho\cos\varphi)\sin\theta \\ x_3 = \rho\sin\varphi. \end{cases}$$

Take the boundary datum $g_\varepsilon = g \in C^1(\partial\Omega,\,\mathscr{N})$ given by

$$g(\mathbf{r}(1,\,\theta,\,\varphi)) = s_*\left\{\left(\mathbf{e}_\theta\cos\frac{\varphi}{2} + \mathbf{e}_\varphi\sin\frac{\varphi}{2}\right)^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right\},$$

where $\mathbf{e}_\theta := \partial_\theta\mathbf{r}/|\partial_\theta\mathbf{r}|$, $\mathbf{e}_\varphi := \partial_\varphi\mathbf{r}$ are orthogonal tangent vectors on the torus. The restriction of g to each slice $\mathbf{r}(\{1\}\times\{\theta\}\times[0,\,2\pi])$ is homotopically non-trivial, so (5.10) is satisfied and $\mathscr{S}_{\mathrm{line}}$ is non-empty. Because of the minimality of Q_ε, we expect $\mathscr{S}_{\mathrm{line}}$ to be length-minimizing among the loops C such that $\mathbf{r}([0,\,1]\times\{\theta\}\times[0,\,2\pi]) \cap C \neq \emptyset$ for all θ. Thus, we *conjecture* that

$$\mathscr{S}_{\mathrm{line}} = \left\{x \in \mathbb{R}^3 \colon x_3 = 0,\ x_1^2 + x_2^2 = 1\right\} \subseteq \partial\Omega.$$

In contrast, for the boundary data constructed in proof of Proposition 5 we expect that $\mathscr{S}_{\mathrm{line}}$ lies inside the domain (more precisely $\mathscr{S}_{\mathrm{line}} = \Omega \cap \{x_1 = x_2 = 0\}$), because of minimality arguments.

5.2 Proof of Proposition 3

If Ω is a bounded, Lipschitz domain and the boundary data are a bounded sequence in $H^{1/2}(\partial\Omega,\,\mathscr{N})$, then the logarithmic bound for minimizers holds as well. We will give now the proof of this fact, by adapting an argument by Rivière (see [53, Proposition 2.1]). Hardt, Kinderlehrer and Lin's re-projection trick (see [34, Proof of Lemma 2.3]) is a key point here.

Proof of Proposition 3. Once again, Lemma 5.1 directly gives the L^∞-bound (5.1), so we only need to prove (5.2) by constructing a suitable comparison function. For any $0 < \varepsilon < 1$, let $u_\varepsilon \in H^1(\Omega,\,\mathbf{S}_0)$ be the harmonic extension of g_ε, i.e. the unique solution of

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Omega \\ u_\varepsilon = g_\varepsilon & \text{on } \partial\Omega. \end{cases}$$

Since $\{g_\varepsilon\}_\varepsilon$ is bounded in $H^{1/2} \cap L^\infty$, the sequence $\{u_\varepsilon\}_\varepsilon$ is bounded in $H^1 \cap L^\infty$. Let $\delta > 0$ be a small parameter to be chosen later. For any $A \in \mathbf{S}_0$ with $|A| \leq \delta$ and any ε, we define

$$u_\varepsilon^A := (\eta_\varepsilon \circ \phi)\,(u_\varepsilon - A)\,\varrho\,(u_\varepsilon - A)$$

where $\phi\colon \mathbf{S}_0 \to \mathbb{R}$ and $\varrho\colon \mathbf{S}_0 \setminus \mathscr{C} \to \mathscr{N}$ are defined respectively in Lemmas 2.5, 2.4, and $\eta_\varepsilon \in C(\mathbb{R}^+,\,\mathbb{R})$ is given by

$$\eta_\varepsilon(r) := \varepsilon^{-1} r \quad \text{if } 0 \leq r < \varepsilon, \qquad \eta_\varepsilon(r) = 1 \quad \text{if } r \geq \varepsilon.$$

By Lemma 2.4 and Corollary 2.10, we have $u_\varepsilon^A \in (H^1 \cap L^\infty)(\Omega, \mathbf{S}_0)$. We differentiate u_ε^A and, taking advantage of the Lipschitz continuity of ϕ (Lemma 2.5), we deduce

$$\left|\nabla u_\varepsilon^A\right|^2 \leq C\left\{(\eta_\varepsilon' \circ \phi)^2 (u_\varepsilon - A) \left|\nabla u_\varepsilon\right|^2 + (\eta_\varepsilon \circ \phi)^2 (u_\varepsilon - A) \left|\nabla \left(\varrho(u_\varepsilon - A)\right)\right|^2\right\}.$$

We apply Corollary 2.10 to bound the derivative of $\varrho(u_\varepsilon - A)$:

$$\left|\nabla u_\varepsilon^A\right|^2 \leq C\left\{(\eta_\varepsilon' \circ \phi)^2 (u_\varepsilon - A) + \frac{(\eta_\varepsilon \circ \phi)^2 (u_\varepsilon - A)}{\phi^2(u_\varepsilon - A)}\right\} \left|\nabla u_\varepsilon\right|^2.$$

On the other hand, there holds

$$f\left(u_\varepsilon^A\right) \leq C \mathbb{1}_{\{\phi(u_\varepsilon - A) \leq \varepsilon\}},$$

so

$$E_\varepsilon\left(u_\varepsilon^A\right) \leq C \int_\Omega \left\{\left(\frac{\mathbb{1}_{\{\phi(u_\varepsilon - A) \geq \varepsilon\}}}{\phi^2(u_\varepsilon - A)} + \varepsilon^{-2} \mathbb{1}_{\{\phi(u_\varepsilon - A) \leq \varepsilon\}}\right) \left|\nabla u_\varepsilon\right|^2 + \varepsilon^{-2} \mathbb{1}_{\{\phi(u_\varepsilon - A) \leq \varepsilon\}}\right\}. \tag{5.11}$$

Now, fix a bounded subset $K \subseteq \mathbf{S}_0$, so large that $u_\varepsilon(x) + B_\delta^{\mathbf{S}_0} \subseteq K$ for a.e. $x \in \Omega$ and any ε (we denote by $B_\delta^{\mathbf{S}_0}$ the set of $Q \in \mathbf{S}_0$ with $|Q| \leq \delta$). We set $K_\varepsilon := K \cap \{\phi \leq \varepsilon\}$. We integrate (5.11) with respect to $A \in B_\delta^{\mathbf{S}_0}$. We apply Fubini-Tonelli theorem and introduce the new variable $B := u_\varepsilon(x) - A$. We obtain

$$\int_{B_\delta^{\mathbf{S}_0}} E_\varepsilon\left(u_\varepsilon^A\right) \mathrm{d}\mathscr{H}^5(A) \leq C \int_\Omega \left\{\left(\int_{K \setminus K_\varepsilon} \frac{\mathrm{d}\mathscr{H}^5(B)}{\phi^2(B)} + \varepsilon^{-2} \mathscr{H}^5(K_\varepsilon)\right) \left|\nabla u_\varepsilon\right|^2 + \varepsilon^{-2} \mathscr{H}^5(K_\varepsilon)\right\} \mathrm{d}x.$$

We claim that

$$\mathscr{H}^5(K_\varepsilon) \leq C\varepsilon^2 \qquad \text{and} \qquad \int_{K \setminus K_\varepsilon} \frac{\mathrm{d}\mathscr{H}^5(B)}{\phi^2(B)} \leq C\left(|\log \varepsilon| + 1\right). \tag{5.12}$$

To simplify the presentation, we postpone the proof of this claim. With the help of (5.12), we obtain

$$\int_{B_\delta^5} E_\varepsilon\left(u_\varepsilon^A\right) \mathrm{d}\mathscr{H}^5(A) \leq C\left\{\left(|\log \varepsilon| + 1\right) \left\|\nabla u_\varepsilon\right\|_{L^2(\Omega)}^2 + 1\right\} \leq C\left(|\log \varepsilon| + 1\right).$$

Therefore, we can choose $A_0 \in \mathbf{S}_0$ such that $|A_0| \leq \delta$ and

$$E_\varepsilon\left(u_\varepsilon^{A_0}\right) \leq C\left(|\log \varepsilon| + 1\right). \tag{5.13}$$

The map $u_\varepsilon^{A_0}$ satisfies the desired energy estimate, but it does not satisfy the boundary condition, since

$$u_\varepsilon^{A_0} = \varrho\left(g_\varepsilon - A_0\right) \qquad \text{on } \partial\Omega. \tag{5.14}$$

To correct this, we consider the maps $\{\varrho_A\}_{A \in B_\delta^{\mathbf{S}_0}}$ defined by

$$\varrho_A \colon Q \in \mathscr{N} \mapsto \varrho(Q - A).$$

This is a continuous family of mappings in $C^1(\mathscr{N}, \mathscr{N})$ and $\varrho_0 = \mathrm{Id}_{\mathscr{N}}$. Therefore, we can choose δ so small that the map $\varrho_A \colon \mathscr{N} \to \mathscr{N}$ is a diffeomorphism for any $A \in B_\delta^{\mathbf{S}_0}$ (in particular for $A = A_0$). Let $\mathscr{U}$ be the set defined by

$$\mathscr{U} := \left\{\lambda Q \colon \lambda \in \mathbb{R}^+,\ Q \in \mathscr{N}\right\}.$$

We extend $\varrho_{A_0}^{-1}$ to a Lipschitz function $\mathscr{F}\colon \mathscr{U} \to \mathscr{U}$ by setting

$$\mathscr{F}(\lambda Q) := \lambda \varrho_{A_0}^{-1}(Q) \qquad \text{for any } \lambda \in \mathbb{R}^+, \ Q \in \mathscr{N}.$$

Remark that any $P \in \mathscr{U} \setminus \{0\}$ can be uniquely written in the form $P = \lambda Q$ for $\lambda \in \mathbb{R}^+$ and $Q \in \mathscr{N}$, so $\mathscr{F}$ is well-defined. Also, $f \circ \mathscr{F}(P) = f(P)$ because $\mathscr{F}(P)$ and P have the same scalar invariants. The map $P_\varepsilon := \mathscr{F} \circ u_\varepsilon^{A_0}$ is well-defined, because $u_\varepsilon^{A_0} \in \mathscr{U}$. Moreover, P_ε belongs to $H^1_{g_\varepsilon}(\Omega, \mathbf{S}_0)$ thanks to (5.14), and satisfies

$$E_\varepsilon(P_\varepsilon) \leq C\left(|\log \varepsilon| + 1\right)$$

due to (5.13). By comparison, the minimizers satisfy (5.2). □

The claim (5.12) follows by this

Lemma 5.4. *For any $R > 0$, there exist positive constants C_R, M_R such that, for any non increasing, non negative function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$, there holds*

$$\int_{B_R^{\mathbf{S}_0}} (g \circ \phi)(Q)\, \mathrm{d}\mathscr{H}^5(Q) \leq C_R \int_0^{M_R} \left(s + s^4\right) g(s)\, \mathrm{d}s.$$

Assuming that the lemma holds true, choose R so large that $K \subseteq B_R^{\mathbf{S}_0}$. Then, the two assertions of Claim (5.12) follow by taking $g = \mathbb{1}_{(0,\,\varepsilon)}$ and $g(s) = \varepsilon^{-2}\mathbb{1}_{(0,\,\varepsilon)}(s) + s^{-2}\mathbb{1}_{[\varepsilon,\,+\infty)}(s)$, respectively. For the sake of clarity, we split the proof of Lemma 5.4 into a few steps. For $r > 0$, we let dist_r denote the geodesic distance in $\partial B_r^{\mathbf{S}_0}$, that is

$$\mathrm{dist}_r(Q,\, A) := \inf\left\{\int_0^1 |\gamma'(t)|\, \mathrm{d}t \colon \gamma \in C^1([0,\,1],\, \partial B_1^{\mathbf{S}_0}),\ \gamma(0) = Q,\ \gamma(1) \in A\right\} \tag{5.15}$$

for any $Q \in \partial B_r^{\mathbf{S}_0}$ and $A \subseteq \partial B_r^{\mathbf{S}_0}$, and set $\mathscr{N}_r' := \mathscr{C} \cap \partial B_r^{\mathbf{S}_0}$. Notice that there exists a positive constant C such that

$$\mathrm{dist}_{|Q|}(Q,\, P) \leq C\, |Q - P| \tag{5.16}$$

for any $Q,\, P \in \mathbf{S}_0$ with $|Q| = |P|$.

Lemma 5.5. *There exists a positive constant α such that*

$$\phi(Q) \geq \alpha\, \mathrm{dist}_{|Q|}\left(Q,\, \mathscr{N}_{|Q|}'\right) \qquad \text{for any } Q \in \mathbf{S}_0.$$

Proof. Fix $Q \in \mathbf{S}_0$ and $P \in \mathscr{N}_{|Q|}'$. By Lemmas 2.1 and 2.4, we can write

$$Q = s\left(\mathbf{n}^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right) + sr\left(\mathbf{m}^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right) \qquad \text{and} \qquad P = -s'\left(\mathbf{p}^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right),$$

for some orthonormal pair $(\mathbf{n},\, \mathbf{m})$, some unit vector $\mathbf{p}$, s, some positive numbers s, s' and $0 \leq r \leq 1$. Through simple algebra, we obtain

$$|Q - P|^2 = \frac{2}{3}s^2\left(r^2 - r + 1\right) - \frac{2}{3}ss'(1 - r) + \frac{2}{3}s'^2 + 2ss'\left\{(\mathbf{n}\cdot\mathbf{p})^2 + r(\mathbf{m}\cdot\mathbf{p})^2\right\}. \tag{5.17}$$

By setting $s' = 0$ and $s = 0$ in this identity, we see that $|P| = |Q|$ if and only if

$$s'^2 = s^2\left(r^2 - r + 1\right). \tag{5.18}$$

By minimizing (with respect to s', $\mathbf{p}$) the right-hand side in (5.17), subject to the constraint (5.18), we find

$$\operatorname{dist}^2(Q,\,\mathscr{N}_1') = \frac{2}{3}s^2\sqrt{r^2-r+1}\left\{(1-r)^2 - \left(\sqrt{r^2-r+1}-1\right)^2\right\} \leq \frac{2}{3}s^2(1-r)^2 = \frac{2}{3}s_*^2\phi^2(Q).$$

Combining this inequality with (5.16), the lemma follows. □

Lemma 5.6. *Let $\mathscr{N}'$ be a compact n-submanifold of a smooth Riemann m-manifold $\mathscr{M}$, and let*

$$U_\delta := \{x \in \mathscr{M} \colon\ \operatorname{dist}_{\mathscr{M}}(x,\,\mathscr{N}') \leq \delta\}$$

be the δ-neighborhood of $\mathscr{N}'$ in $\mathscr{M}$, for $\delta > 0$ (here $\operatorname{dist}_{\mathscr{M}}$ stands for the geodesic distance in $\mathscr{M}$). There exist $\delta_* > 0$ and, for any $\delta \in (0,\,\delta_*)$, a constant $C = C(\mathscr{M},\,\mathscr{N}',\delta) > 0$ such that for any decreasing function $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ there holds

$$\int_{U_\delta} h\left(\operatorname{dist}_{\mathscr{M}}(x,\,\mathscr{N}')\right)\,\mathrm{d}\mathscr{H}^m(x) \leq C\int_0^{C\delta} s^{m-n-1}h(s)\,\mathrm{d}s.$$

Proof. We identify $\mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n}$, and call the variable $y = (y',\,z) \in \mathbb{R}^n \times \mathbb{R}^{m-n}$. For a small $\delta_* > 0$, the δ_*-neighborhood U_{δ_*} can be covered with finitely many open sets $\{V_j\}_{1\leq j\leq K}$ and, for each j, there exists a bilipschitz homeomorphism $\varphi_j\colon V_j \to W_j \subseteq \mathbb{R}^m$ which maps $\mathscr{N}' \cap V_j$ onto $\mathbb{R}^n \cap W_j$. Due to the bilipschitz continuity of the φ_j's, there exist two constants γ_1, γ_2 such that, for any j and any $y = (y',\,z) \in W_j$, there holds

$$\gamma_1\,|z| \leq \operatorname{dist}_{\mathscr{M}}(\varphi_j^{-1}(y),\,\mathscr{N}) \leq \gamma_2\,|z|\,.$$

Therefore, if $0 < \delta < \delta_*$ the change of variable $x = \varphi_j^{-1}(y)$ implies

$$\begin{aligned}\int_{U_\delta} h\left(\operatorname{dist}_{\mathscr{M}}(x,\,\mathscr{N}')\right)\,\mathrm{d}\mathscr{H}^m(x) &\leq \sum_{j=1}^K \int_{\varphi_j^{-1}(V_j)} h\left(\gamma_1|z|\right)\,\left|\mathrm{J}\varphi_j^{-1}(y)\right|\,\mathrm{d}\mathscr{H}^m(y)\\ &\leq M\int_{B^{m-n}(0,\,\gamma_2\delta)} h(\gamma_1|z|)\,\mathrm{d}\mathscr{H}^{m-n}(z)\end{aligned}$$

where M is an upper bound for the norm of the Jacobians $\mathrm{J}\varphi_j^{-1}$. Then, passing to polar coordinates,

$$\begin{aligned}\int_{U_\delta} h\left(\operatorname{dist}_{\mathscr{M}}(x,\,\mathscr{N}')\right)\,\mathrm{d}\mathscr{H}^m(x) &\leq M\int_0^{\gamma_2\delta} \rho^{m-n-1}h(\gamma_1\rho)\,\mathrm{d}\rho\\ &\leq M\gamma_1^{1+n-m}\int_0^{\gamma_1\gamma_2\rho} s^{m-n-1}h(s)\,\mathrm{d}s.\end{aligned}$$ □

Proof of Lemma 5.4. By Lemma 2.5, the function ϕ is positively homogeneous of degree 1. Then,

$$\int_{B_R^{\mathbf{s}_0}} (g\circ\phi)(Q)\,\mathrm{d}\mathscr{H}^5(Q) = \int_0^R \rho^4\int_{\partial B_1^{\mathbf{s}_0}} g\left(\rho\phi(Q)\right)\,\mathrm{d}\mathscr{H}^4(Q)\,\mathrm{d}\rho.$$

By applying Lemma 5.5, and since g is a decreasing function,

$$\int_{B_R^{\mathbf{s}_0}} (g\circ\phi)(Q)\,\mathrm{d}\mathscr{H}^5(Q) \leq \int_0^R \rho^4\int_{\partial B_1^{\mathbf{s}_0}} g\left(\alpha\rho\operatorname{dist}_1(Q,\,\mathscr{N}_1')\right)\,\mathrm{d}\mathscr{H}^4(Q)\,\mathrm{d}\rho.$$

Now, we apply Lemma 5.6 with $\mathscr{M} = \partial B_1^{\mathbf{S}_0}$, $\mathscr{N}' = \mathscr{N}_1'$ and $h\colon s \mapsto g(\alpha\rho s)$. We find constants δ and C such that, letting U_δ be the δ-neighborhood of $\mathscr{N}_1'$ in $\partial B_1^{\mathbf{S}_0}$ and $V_\delta := \partial B_1^{\mathbf{S}_0} \setminus U_\delta$, we have

$$\begin{aligned}
&\int_{B_R^{\mathbf{S}_0}} (g\circ\phi)(Q)\,\mathrm{d}\mathscr{H}^5(Q) \\
&\qquad = \int_0^R \rho^4 \left\{\int_{U_\delta} g\left(\alpha\rho\,\mathrm{dist}_1(Q,\,\mathscr{N}_1')\right)\mathrm{d}\mathscr{H}^4(Q) + \int_{V_\delta} g\left(\alpha\rho\,\mathrm{dist}_1(Q,\,\mathscr{N}_1')\right)\mathrm{d}\mathscr{H}^4(Q)\right\}\mathrm{d}\rho \\
&\qquad \leq C\int_0^R \rho^4\left\{\int_0^{C\delta} sg(\alpha\rho s)\,\mathrm{d}s + g(\alpha\rho\delta)\,\mathscr{H}^4(V_\delta)\right\}\mathrm{d}\rho
\end{aligned}$$

(to bound the integral on V_δ, we use again that g is decreasing). Now, the two terms can be easily handled by changing the variables and using Fubini-Tonelli theorem:

$$\begin{aligned}
\int_{B_R^{\mathbf{S}_0}} (g\circ\phi)(Q)\,\mathrm{d}\mathscr{H}^5(Q) &\leq \alpha^{-2}C\int_0^R \rho^2\int_0^{\alpha\rho\delta C} tg(t)\,\mathrm{d}t\,\mathrm{d}\rho + (\alpha\delta)^{-5}C\mathscr{H}^4(V_\delta)\int_0^{\alpha\delta R} t^4 g(t)\,\mathrm{d}t \\
&\leq C_{\alpha,\delta,R}\int_0^{C_{\alpha,\delta,R}} \left(t+t^4\right) g(t)\,\mathrm{d}t.
\end{aligned}$$

Since α, δ depend only on ϕ and $\mathscr{N}_1'$, the lemma is proved. $\square$

6 Coexistence of line and point singularities: an example

In this section, we show through an example that both the set of line singularities $\mathscr{S}_{\mathrm{line}}$ and the set of point singularities $\mathscr{S}_{\mathrm{pts}}$ may be non-empty. We consider the following domain. For two fixed positive numbers

$$L > 0 \qquad \text{and} \qquad 0 < r < \frac{1}{2}, \tag{6.1}$$

define

$$p_\pm := (\pm(L+1),\,0,\,0), \qquad \Omega_\pm := B_1(p_\pm), \qquad \Omega_0 := \left([-L-1,\,L+1]\times B^2_r(0)\right)\setminus(\Omega_-\cup\Omega_+)$$

and $\Omega := \Omega_-\cup\Omega_0\cup\Omega_+$. In other words, the domain consists of two balls joined by a cylinder about the x_1-axis (see Figure 1). This is a Lipschitz domain; however, one could consider a smooth domain Ω', obtained from Ω by "smoothing the corners", and the results of this section could be easily adapted to Ω'.

In order to construct the boundary datum, we write $\partial\Omega = \Gamma_-\cup\Gamma_0\cup\Gamma_+$, where $\Gamma_\pm := \partial\Omega_\pm\setminus\Omega_0$ and $\Gamma_0 := \partial\Omega_0\setminus(\overline{\Omega}_+\cup\overline{\Omega}_-)$. We define the auxiliary functions $\chi\in H^1(0,\,2\pi)$, $\eta_\varepsilon\in H^1(0,\,\pi)$ and $\xi_r\in H^1(0,\,\pi)$ by

$$\chi(\theta) := \begin{cases} \pi/2 - 3\theta/5 & \text{for } 0\leq\theta\leq 5\pi/6 \\ 0 & \text{for } 5\pi/6<\theta<7\pi/6 \\ 7\pi/10 - 3\theta/5 & \text{for } 7\pi/6\leq\theta\leq 2\pi, \end{cases} \qquad \eta_\varepsilon(\varphi) := \begin{cases} \varepsilon^{-1}\varphi & \text{for } 0\leq\varphi\leq\varepsilon \\ 1 & \text{for } \varepsilon<\varphi<\pi-\varepsilon \\ \varepsilon^{-1}(\pi-\varphi) & \text{for } \pi-\varepsilon\leq\varphi\leq\pi \end{cases}$$

and

$$\xi_r(\tilde\varphi) := \begin{cases} 0 & \text{if } 0\leq\tilde\varphi\leq\arcsin r \\ \dfrac{\arcsin 2r}{\arcsin 2r - \arcsin r}\,(\tilde\varphi - \arcsin r) & \text{if } \arcsin 2r<\tilde\varphi<\arcsin 2r \\ \tilde\varphi & \text{if } \arcsin 2r\leq\tilde\varphi\leq\pi. \end{cases}$$

Notice that

$$|\eta_\varepsilon'| \leq \varepsilon^{-1} \qquad \text{and} \qquad |\xi_r'| \leq 2. \tag{6.2}$$

The boundary datum g_ε is defined as follows. We parametrize Γ_+ using spherical coordinates $(\theta, \varphi) \in [0, 2\pi] \times [0, \pi]$ centered at p_+:

$$x_1 = L + 1 + \sin\varphi\cos\theta, \qquad x_2 = \sin\varphi\sin\theta, \qquad x_3 = \cos\varphi$$

and define g_ε on Γ_+ by

$$g_\varepsilon\colon (x_1, x_2, x_3) \in \Gamma_+ \mapsto s_*\eta_\varepsilon(\varphi)\left\{\Big(\mathbf{e}_1\cos\chi(\theta) + \mathbf{e}_2\sin\chi(\theta)\Big)^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right\}. \tag{6.3}$$

On Γ_-, we use the coordinates $(\tilde\theta, \tilde\varphi) \in [0, 2\pi] \times [0, \pi]$ given by

$$x_1 = -L - 1 + \cos\tilde\varphi, \qquad x_2 = \sin\tilde\varphi\cos\tilde\theta, \qquad x_3 = \sin\tilde\varphi\sin\tilde\theta$$

and set

$$g_\varepsilon\colon (x_1, x_2, x_3) \in \Gamma_- \mapsto s_*\left\{\Big(\mathbf{e}_1\cos\xi_r(\tilde\varphi) + \mathbf{e}_2\sin\xi_r(\tilde\varphi)\cos\tilde\theta + \mathbf{e}_3\sin\xi_r(\tilde\varphi)\sin\tilde\theta\Big)^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right\}. \tag{6.4}$$

Finally, we set

$$g_\varepsilon := s_*\left(\mathbf{e}_1^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right) \qquad \text{on } \Gamma_0.$$

The boundary datum is non-orientable and has two point singularities on Γ_+, is constant on Γ_0 and has a hedgehog-type behaviour on Γ_-. In Figure 1, we represent the direction of the eigenvector associated with the leading eigenvalue of $g_\varepsilon(x)$, for $x \in \partial\Omega$. Of course, one could regularize the functions χ, η_ε and ξ_r so that the map g_ε is smooth; this would not affect our arguments. As usual, pick a subsequence ε_n so that the measures μ_{ε_n} defined by (4.17) converge weakly* in $C(\overline{\Omega})'$ to a measure μ_0, and let $\mathscr{S}_{\mathrm{line}} \subseteq \overline{\Omega}$ be the support of μ_0. Let $\mathscr{S}_{\mathrm{pts}} \subseteq \Omega \setminus \mathscr{S}_{\mathrm{line}}$ be a set such that the sequence $\{Q_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is compact in $C^0(\Omega \setminus (\mathscr{S}_{\mathrm{line}} \cup \mathscr{S}_{\mathrm{pts}}), \mathbf{S}_0)$. By Theorem 1, such a set exists and is locally finite in $\Omega \setminus \mathscr{S}_{\mathrm{line}}$. We will show the following result, which implies Proposition 6.

Proposition 6.1. *There exists L^* such that, if*

$$L \geq L^*$$

then $\emptyset \neq \mathscr{S}_{\mathrm{line}} \subseteq \{x_1 \geq 0\}$ and $\mathscr{S}_{\mathrm{pts}} \cap \{x_1 \leq -L/2\} \neq \emptyset$.

Remark 6.1. The presence of a point defect is *not* forced by a topological obstruction. In other words, there exists maps $P_\varepsilon\colon \Omega \to \mathbf{S}_0$ which satisfy $P_\varepsilon = g_\varepsilon$ on $\partial\Omega$ and converge to a map with a line singularity but no point singularity. Indeed, let $\varphi\colon \overline{\Omega} \to \overline{B}_1$ be a bilipschitz homeomorphism such that $\varphi(L, 0, \pm 1) = (0, 0, \pm 1)$. Then, the functions

$$P_\varepsilon(x) := g_\varepsilon \circ \varphi^{-1}\left(\frac{\varphi(x)}{|\varphi(x)|}\right)$$

converge a.e. to a map with a line singularity $\mathscr{S}_{\mathrm{line}} := \varphi^{-1}\{x_1 = x_2 = 0\}$, but no point singularities. The convergence also holds in $H^1_{\mathrm{loc}}(\Omega \setminus \mathscr{S}_{\mathrm{line}}, \mathbf{S}_0)$.

We split the proof of Proposition 6.1 into some lemmas. Throughout the section, we use the symbol C to denote a generic constant, which does not depend on ε, L and r.

Lemma 6.2. *There exists a constant M,* independent of L and r, *such that*

$$E_\varepsilon(Q_\varepsilon,\,\Omega) \leq M\left(|\log\varepsilon| + 1\right) \qquad \textit{and} \qquad \|Q_\varepsilon\|_{L^\infty(\Omega)} \leq M$$

for any $0 < \varepsilon < 1$.

Proof. The L^∞-bound follows by Lemma 5.1, since $|g_\varepsilon(x)| \leq (2/3)^{1/2} s_*$ for a.e. $x \in \partial\Omega$ and any $0 < \varepsilon < 1$. The energy bound follows by a comparison argument. We define a map P_ε on Ω_+ and Ω_- by homogeneous extension:

$$P_\varepsilon(x) := g_\varepsilon\left(p_+ + \frac{x - p_+}{|x - p_+|}\right) \ \text{ if } x \in \Omega_+, \qquad P_\varepsilon(x) := g_\varepsilon\left(p_- + \frac{x - p_-}{|x - p_-|}\right) \ \text{ if } x \in \Omega_-,$$

whereas we set

$$P_\varepsilon(x) := s_*\left(\mathbf{e}_1^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right) \ \text{ if } x \in \Omega_0.$$

(Here we assume that g_ε is defined also on $\partial\Omega_\pm \setminus \Gamma_\pm$, by the same formulae (6.3)–(6.4).) Then, the map P_ε is continuous and belongs to $H^1(\Omega,\,\mathbf{S}_0)$. Moreover, $E_\varepsilon(P_\varepsilon,\,\Omega_0) = 0$ since $P_{\varepsilon|\Omega_0}$ is constant and takes values in $\mathscr{N}$. A simple computation, based on (6.2), concludes the proof. □

For any $s \in \mathbb{R}$, let $U_s := \Omega \cap \{x_1 < s\}$ and $G_\varepsilon(s) := E_\varepsilon(Q_\varepsilon,\,U_s)$. Fubini-Tonelli theorem entails that $G'_\varepsilon(s) = E_\varepsilon(Q_\varepsilon,\,\Omega \cap \{x_1 = s\})$ for a.e. s.

Lemma 6.3. *There exist positive constants L_* and M such that, for any $L \geq L_*$ and any $0 < \varepsilon \leq 1/2$, there holds*

$$G_\varepsilon(0) \leq M.$$

In particular, if $L \geq L_*$ then $\mathscr{S}_{\mathrm{line}} \subseteq \overline{\Omega} \setminus U_0$.

Proof. This proof is based on the same arguments as Proposition 7. Define the set

$$D^\varepsilon := \left\{s \in (0,\,L)\colon G'_\varepsilon(s) \leq \frac{2M}{L}\left(|\log\varepsilon| + 1\right)\right\}.$$

By Lemma 6.2 and an average argument, we know that

$$\mathscr{H}^1(D^\varepsilon) \geq \frac{L}{2}. \tag{6.5}$$

Moreover, there exists $L_* > 0$ such that, for any $L \geq L_*$, any $0 < \varepsilon \leq 1/2$ and any $s \in D^\varepsilon$, there holds

$$G'_\varepsilon(s) \leq \eta_0\,|\log\varepsilon|$$

where η_0 is the constant given by Proposition 3.6. Therefore, for a fixed $s \in D^\varepsilon$ we can apply Proposition 3.6 to the map $u_\varepsilon := Q_{\varepsilon|\{s\}\times B^2_r}$. Notice that u_ε is defined on a disk, whereas the maps we consider in Proposition 3.6 are defined over a sphere. However, since u_ε takes a constant value on the boundary, it can be identified with a map defined on a sphere by collapsing $\{s\} \times \partial B^2_r$ into a point. Setting $h(\varepsilon) := \varepsilon^{1/2}|\log\varepsilon|$ and $A_\varepsilon := (s - h(\varepsilon),\,s) \times B^2_r$, we find maps $v_\varepsilon\colon \{s - h(\varepsilon)\} \times B^2_r \to \mathscr{N}$ and $\varphi_\varepsilon\colon A_\varepsilon \to \mathbf{S}_0$ such that

$$\frac{1}{2}\int_{\{s-h(\varepsilon)\}\times B^2_r} |\nabla v_\varepsilon|^2 \,\mathrm{d}\mathscr{H}^2 \leq G'_\varepsilon(s), \qquad E_\varepsilon(\varphi_\varepsilon) \leq Ch(\varepsilon)\,|\log\varepsilon|\,. \tag{6.6}$$

Now, consider the set $V_s := [s-h(\varepsilon)-r,\, s-h(\varepsilon)]\times B^2_r$ (we assume that $L_* > 2$, so that $s-h(\varepsilon)-r > -L$ for $\varepsilon \leq 1/2$) and the map $\tilde{v}_\varepsilon \in H^1(\partial V_s,\, \mathscr{N})$ given by $\tilde{v}_\varepsilon := v_\varepsilon$ on $\{s\}\times B^2_r$,

$$\tilde{v}_\varepsilon := s_*\left(\mathbf{e}_1^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right) \qquad \text{on } \partial V_s \setminus \left(\{s\}\times B^2_r\right).$$

Thanks to (6.6), we have

$$\frac{1}{2}\int_{\partial V_s} |\nabla \tilde{v}_\varepsilon|^2 \,\mathrm{d}\mathscr{H}^2 = \frac{1}{2}\int_{\{s\}\times B^2_r} |\nabla v_\varepsilon|^2 \,\mathrm{d}\mathscr{H}^2 \leq G'_\varepsilon(s).$$

Then, by applying Lemma 3.1 (which is possible because V_s is bilipschitz equivalent to a ball), we find a map $w_\varepsilon \in H^1(V_s,\, \mathscr{N})$ such that $w_\varepsilon = \tilde{v}_\varepsilon$ on ∂V_s and

$$\frac{1}{2}\int_{V_s} |\nabla w_\varepsilon|^2 \leq C G'_\varepsilon(s)^{1/2}, \tag{6.7}$$

for a constant C independent of ε, L, r. (Here we have used that $r < 1$.) Finally, we define a map $\tilde{w}_\varepsilon \in H^1(U_s,\, \mathbf{S}_0)$ as follows. We set $\tilde{w}_\varepsilon := \varphi_\varepsilon$ on $[s-h(\varepsilon),\, s]\times B^2_r$ and $\tilde{w}_\varepsilon := w_\varepsilon$ on V_s,

$$\tilde{w}_\varepsilon := s_*\left(\mathbf{e}_1^{\otimes 2} - \frac{1}{3}\,\mathrm{Id}\right) \qquad \text{on } \Omega_0 \setminus \left([s-h(\varepsilon)-r,\, s]\times B^2_r\right)$$

and use an homogeneous extension to construct $\tilde{w}_\varepsilon$ on Ω_-:

$$\tilde{w}_\varepsilon(x) := g_\varepsilon\left(p_- + \frac{x-p_-}{|x-p_-|}\right) \qquad \text{for } x\in\Omega_-.$$

The map $\tilde{w}_\varepsilon$ is continuous, satisfies $E_\varepsilon(\tilde{w}_\varepsilon) = 0$ on $\Omega_0 \setminus ([s-h(\varepsilon)-r,\, s]\times B^2_r)$ and $E_\varepsilon(\tilde{w}_\varepsilon,\, \Omega_-) \leq C$ because of (6.2). Thus, from (6.6) and (6.7) we infer

$$E_\varepsilon(\tilde{w}_\varepsilon,\, U_s) \leq C G'_\varepsilon(s)^{1/2} + C.$$

Moreover, $\tilde{w}_\varepsilon$ is an admissible competitor for Q_ε, because $\tilde{w}_\varepsilon = Q_\varepsilon$ on ∂U_s. Then, the minimality of Q_ε yields

$$G_\varepsilon(s) \leq C G'_\varepsilon(s)^{1/2} + C \qquad \text{for a.e. } s\in D^\varepsilon \text{ and every } 0<\varepsilon\leq\frac{1}{2}. \tag{6.8}$$

Thanks to (6.5) and (6.8), we apply Lemma 4.2 to $y := G_\varepsilon$ and obtain

$$G_\varepsilon(0) \leq \left(1+\frac{2}{L}\right)C \leq \left(1+\frac{2}{L_*}\right)C =: M.$$

for every $0<\varepsilon\leq 1/2$ Therefore, $\mu_\varepsilon \llcorner U_0 \to 0$ in $\mathscr{M}_{\mathrm{b}}(U_0) := C_0(U_0)'$ and hence $\mathscr{S}_{\mathrm{line}} \subseteq \overline{\Omega}\setminus U_0$. □

Before concluding the proof of Proposition 6.1, we recall a well-known fact. Given $r>0$ and a continuous maps $\mathbf{n}\colon B^2_r \to \mathbb{S}^2$ which take a constant value on ∂B^2_r, it is possible to define the topological degree of $\mathbf{n}$. Indeed, the topological space which is obtained by collapsing the boundary of ∂B^2_r into a point is homeomorphic to a sphere. Then, since $\mathbf{n}_{|\partial B^2_r}$ is a constant, $\mathbf{n}$ induces a continuous map $\mathbb{S}^2\to\mathbb{S}^2$ whose homotopy class is characterized by an integer number d called the degree of $\mathbf{n}$. We will write $d = \deg(\mathbf{n},\, B^2_r)$. In case $\mathbf{n}\in H^1(B^2_r,\, \mathbb{S}^2)$ takes a constant value at the boundary, the degree of $\mathbf{n}$ can still be defined (for instance, one can apply the VMO-theory by Brezis and Nirenberg [19, 20]).

Lemma 6.4. *For any $r > 0$ and any $\mathbf{n} \in H^1(B_r^2, \mathbb{S}^2)$ with constant value at the boundary, if*

$$\frac{1}{2}\int_{B_r^2} |\nabla \mathbf{n}|^2 \, \mathrm{d}\mathscr{H}^2 < 4\pi$$

then $\deg(\mathbf{n}, B_r^2) = 0$.

Proof. By applying the area formula, we obtain

$$\int_{B_r^2} |\partial_{x_1}\mathbf{n} \times \partial_{x_2}\mathbf{n}| \, \mathrm{d}\mathscr{H}^2 = \int_{\mathbf{n}(B_r^2)} \mathscr{H}^0\left(\mathbf{n}^{-1}(y)\right) \mathrm{d}\mathscr{H}^2(y) \geq \mathscr{H}^2\left(\mathbf{n}(B_r^2)\right).$$

On the other hand, we have $|\partial_{x_1}\mathbf{n} \times \partial_{x_2}\mathbf{n}| \leq |\partial_{x_1}\mathbf{n}||\partial_{x_2}\mathbf{n}| \leq |\nabla\mathbf{n}|^2/2$. Therefore, there holds

$$\frac{1}{2}\int_{B_r^2} |\nabla \mathbf{n}|^2 \, \mathrm{d}\mathscr{H}^2 \geq \mathscr{H}^2\left(\mathbf{n}(B_r^2)\right).$$

If the left-hand side is $< 4\pi$, then $\mathbf{n}$ is not surjective and so $\deg(\mathbf{n}, B_r^2) = 0$ (see e.g. [19, Property 1]). □

Proof of Proposition 6.1. Arguing as in the proof of Proposition 5, and using that the boundary conditions $g_{\varepsilon|U_+}$ are non orientable, one shows that

$$E_\varepsilon(Q_\varepsilon) \geq C\left(|\log\varepsilon| - 1\right)$$

for any ε, L and r, so $\mathscr{S}_{\mathrm{line}} \neq \emptyset$. By Lemma 6.3, there exists L_* such that $\mathscr{S}_{\mathrm{line}} \subseteq \Omega \setminus U_0$ if $L \geq L_*$. Set

$$L^* := \max\left\{L_*, \, \frac{M}{\pi s_*}\right\} \tag{6.9}$$

where M is given by Lemma 6.3, and let $L \geq L^*$. The proposition follows once we show that $\mathscr{S}_{\mathrm{pts}} \cap U_{-L/2} \neq \emptyset$.

By applying Lemma 6.3, Theorem 1 and Corollary 4.12, we deduce that $Q_{\varepsilon_n} \to Q_0$ in $H^1(U_{-\delta}, \mathbf{S}_0)$ for every $\delta > 0$, where $Q_0 \colon U_0 \to \mathscr{N}$ is a locally minimizing harmonic map. Passing to the limit as $\varepsilon \to 0$ in Lemma 6.3, we see that

$$\frac{1}{2}\int_{U_0} |\nabla Q_0|^2 \leq M \tag{6.10}$$

for an L-independent constant M. In particular, $Q_0 \in H^1(U_0, \mathscr{N})$. An average argument, combined with Lemma 6.10, shows that there exists $-L < s < -L/2$ such that $Q_0 \in H^1(\{s\} \times B_r^2, \mathscr{N})$ and

$$\frac{1}{2}\int_{\{s\}\times B_r^2} |\nabla Q_0|^2 \, \mathrm{d}\mathscr{H}^2 \leq \frac{4M}{L}.$$

Due to Lemma 3.4, we find a lifting of $Q_{0|\partial U_s}$, i.e. a map $\mathbf{n}_0 \in H^1(\partial U_s, \mathbb{S}^2)$ which satisfies (3.1) and $|\nabla Q_0|^2 = 2s_*|\nabla\mathbf{n}_0|^2$ a.e. Then, we have

$$\frac{1}{2}\int_{\{s\}\times B_r^2} |\nabla \mathbf{n}_0|^2 \, \mathrm{d}\mathscr{H}^2 \leq \frac{2M}{s_* L}. \tag{6.11}$$

Combining (6.11) with (6.9), we deduce

$$\frac{1}{2}\int_{\{s\}\times B_r^2} |\nabla \mathbf{n}_0|^2 \, \mathrm{d}\mathscr{H}^2 \leq 2\pi.$$

Moreover, $\mathbf{n}_0$ takes a constant value on the boundary of $\{s\} \times B_r^2$, since Q_0 does. Then, by Lemma 6.4, $\deg(\mathbf{n}_0, \{s\} \times B_r^2) = 0$. On the other hand, $\deg(\mathbf{n}_0, \partial U_s \cap \Gamma_0) = 0$ since $\mathbf{n}_0$ takes a constant value on $\partial U_s \cap \Gamma_0$, and $\deg(\mathbf{n}_0, \Gamma_-)$ can be computed explicitly thanks to (6.4). This yields

$$\deg(\mathbf{n}_0, \partial U_s) = \deg(\mathbf{n}_0, \Gamma_-) = \pm 1,$$

so the map $Q_{0|\partial U_s}$ is homotopically non-trivial and $\mathscr{S}_{\mathrm{pts}} \cap U_s \neq \emptyset$. □

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